1. Generalities on Tensor Products

Write \( \text{Spaces} \) for \( \text{Ch}^- (\text{Shv}_N(\text{Cor}_k)) \). Write \( C \) for the category \( \text{Ch}^- (\text{PST}) \). We consider three different products on \( C \). Suppose \( A^*, B^* \in C \), and that \( P^*, Q^* \in C \) are chain complexes of direct sums of representable elements such that \( P^* \to A^* \) and \( Q^* \to B^* \)
- \( A^* \otimes_Z B^* = \text{Tot}(A^i \otimes_Z B^{n-i}) \)
- \( A^* \otimes_{\text{tr},L} B^* = \text{Tot}(A^i \otimes_{\text{tr}} B^{n-i}) \), where \( \otimes_{\text{tr}} \) is as defined in [MVW06, chapter 8]
- \( A^* \otimes_{\text{tr},L} B^* = \text{Tot}(P^i \otimes_{\text{tr}} Q^{n-i}) \)

**Proposition 1.** Suppose \( g : B \to B' \in C \). Then there are natural transformations

1. \( - \otimes_Z B \to - \otimes_Z B' \).
2. \( - \otimes_{\text{tr},L} B \to - \otimes_{\text{tr},L} B' \).
3. \( - \otimes_{\text{tr}} B \to - \otimes_{\text{tr}} B \).

**Proof.** Both tensor products are left-adjoints, and so commute with direct sums. A routine consideration of diagrams of double-complexes, which we do not typset here, is enough to reduce all three statements to the equivalent statements in \( \text{PST} \).

Suppose now that \( A \to A' \) and \( B \to B' \) are maps in \( \text{PST} \). The following diagram commutes

\[
\begin{array}{ccc}
A \otimes_Z B & \to & A \otimes_Z B' \\
\downarrow & & \downarrow \\
A' \otimes_Z B & \to & A' \otimes_Z B'
\end{array}
\]

since it commutes for abelian groups.

The verification of the same diagram with \( \otimes_{\text{tr},L} \) instead of \( \otimes_Z \) is somewhat more involved. One replaces \( A, A', B, B' \) by chain complexes \( X_*, X'_*, Y_*, Y'_* \) of direct sums of representable elements. The tensor product \( \otimes_{\text{tr}} \) is by construction functorial from the total complexes \( X_* \otimes_{\text{tr}} Y_* \) etc. To verify that the requisite diagram

\[
\begin{array}{ccc}
X_* \otimes_{\text{tr}} Y_* & \to & X_* \otimes_{\text{tr}} Y'_* \\
\downarrow & & \downarrow \\
X'_* \otimes_{\text{tr}} Y_* & \to & X'_* \otimes_{\text{tr}} Y'_*
\end{array}
\]

commutes we can again reduce to checking componentwise, which is simply verifying the commutativity of

\[
\begin{array}{ccc}
X \times Y & \to & X \times Y' \\
\downarrow & & \downarrow \\
X' \times Y & \to & X' \times Y'
\end{array}
\]

for schemes.

Finally, since \( \otimes_Z \) is right-exact, we can compute \( A \otimes_Z B \) as \( H_0(\text{Tot}(X_* \otimes_Z Y_*)) \), which allows us to reduce the final verification to the commutativity of the following diagram for schemes

\[
\begin{array}{ccc}
X \otimes_Z Y & \to & X \otimes_{\text{tr}} Y \\
\downarrow & & \downarrow \\
X' \otimes_Z Y & \to & X' \otimes_{\text{tr}} Y
\end{array}
\]

This follows from the construction in [MVW06]. \( \square \)

**Proposition 2.** In the notation of the previous proposition, there are natural transformations

1. \( - \otimes_{\text{tr},L} B \to - \otimes_{\text{tr},L} B' \).
2. \( - \otimes_{\text{tr},L} B \to - \otimes_{\text{tr}} B \).
Proof. The proofs are routine.

Since sheafification is exact, these results have analogues for \( \otimes_{\text{Nis}} \), \( \otimes_{\text{Nis}}^{\text{tr}} \), \( \otimes_{\text{Nis}}^{\text{tr,L}} \). We recall that the latter is the tensor product which descends to \( \text{DM}_{\text{Nis}}^{\text{eff}} \).

2. Generalities on the Module Structure of Motivic Cohomology

Write \( \text{Spaces} \) for \( \text{Ch}^{+} (\text{Shv}_{\text{Nis}} (\text{Cor}_{k})) \). In this section, we assume that Voevodsky’s cancellation theorem, [MVW06, 16.25], holds; it holds if \( k \) admits resolution of singularities or if \( k \) is perfect.

Proposition 3. Let \( A, B, C, D, E, F \in \text{DM}_{\text{Nis}}^{\text{eff}} \). Let \( \phi : A \to B \) be a map. Let \( I = Z[0](0) = M(\text{pt}) \in \text{DM}_{\text{Nis}}^{\text{eff}} \) be the derived motive of a point. Then the following are true

1. There exists a homomorphism
   \[
   \text{Hom}_{\text{DM}_{\text{Nis}}^{\text{eff}}} (A, B) \otimes \text{Hom}_{\text{DM}_{\text{Nis}}^{\text{eff}}} (C, D) \to \text{Hom}_{\text{DM}_{\text{Nis}}^{\text{eff}}} (A \otimes C, B \otimes D)
   \]
   which is natural in \( A, B, C, D \), and associative in the sense that the following diagram commutes

   \[
   \begin{array}{ccc}
   \text{Hom}(A, C) \otimes \text{Hom}(B, D) \otimes \text{Hom}(E, F) & \to & \text{Hom}(A, C) \otimes \text{Hom}(B \otimes E, D \otimes F) \\
   \downarrow & & \downarrow \\
   \text{Hom}(A \otimes B, C \otimes D) \otimes \text{Hom}(E, F) & \to & \text{Hom}(A \otimes C \otimes E, B \otimes D \otimes F)
   \end{array}
   \]

2. There are isomorphisms \( A \otimes I \cong A \cong I \otimes A \), natural in \( A \), such that the following composition

   \[
   \text{Hom}(I, I) \otimes \text{Hom}(A, B) \to \text{Hom}(I \otimes A, I \otimes B) \to \text{Hom}(A, B)
   \]

   takes the pair \((\text{id}, \phi)\) to \( \phi \).

Proof. The homomorphism \( \text{Hom}_{\text{DM}_{\text{Nis}}^{\text{eff}}} (A, B) \otimes \text{Hom}_{\text{DM}_{\text{Nis}}^{\text{eff}}} (C, D) \to \text{Hom}_{\text{DM}_{\text{Nis}}^{\text{eff}}} (A \otimes C, B \otimes D) \) arises as follows. Suppose \( \phi \in \text{Hom}(A, B), \psi \in \text{Hom}(C, D) \), we form \( \phi \ast \psi \in \text{Hom}(A \otimes C, B \otimes D) \) as

   \[
   A \otimes C \xrightarrow{\phi \otimes \psi} B \otimes C \xrightarrow{B \otimes \psi} B \otimes D
   \]

It is routine to verify this is bilinear, natural and that the associativity diagram holds. The identity isomorphism derives from a diagram

\[
\begin{array}{ccc}
I \otimes A & \xrightarrow{\text{id} \otimes \phi} & I \otimes C \\
\downarrow & \equiv & \downarrow \\
A & \equiv & C
\end{array}
\]

Which is precisely the statement that the isomorphism \( A \otimes I \cong A \) is natural, which can be verified in \( \text{Spaces} \) and carried through to \( \text{DM}_{\text{Nis}}^{\text{eff}} \).

Proposition 4. In \( \text{DM}_{\text{Nis}}^{\text{eff}} \), \( Z[p](q) \otimes Z[p'](q') \cong Z[p + p'](q + q') \)

Proof. It suffices to prove that \( Z(q) \otimes Z(1) \cong Z(q + 1) \). The element \( Z(q) \in \text{DM}_{\text{Nis}}^{\text{eff}} \) can be represented by \( Z_{\text{eff}}(G^{\wedge q}) \in \text{PST} \). To compute \( Z_{\text{eff}}(G^{\wedge q}) \otimes_{\text{tr,L}} Z_{\text{eff}}(G_{m}) \), we use the resolution

\[
\begin{array}{c}
0 \to Z \to \bigoplus_{i \leq 1} Z_{\text{eff}}(G_{m}) \to \bigoplus_{i < j} Z_{\text{eff}}(G_{m} \times G_{m}) \\
\cdots \to \bigoplus_{i} Z_{\text{eff}} \left( \prod_{j \neq i} G_{m} \right) \to Z_{\text{eff}} \left( \prod G_{m} \right) \to Z(G^{\wedge q}) \to 0
\end{array}
\]

The double complex we use to compute \( Z_{\text{eff}}(G^{\wedge q}) \otimes_{\text{tr,L}} Z_{\text{eff}}(G_{m}) \) is nonzero only in two rows, where the typical square is of the form

\[
\bigoplus_{|I|=k} Z_{\text{eff}} \left( \prod_{i \in I} G_{m} \right) \times Z \to \bigoplus_{|I|=k} Z_{\text{eff}} \left( \prod_{i \in I} G_{m} \right) \times Z
\]

Upon taking total complexes, we have the corresponding resolution for \( Z_{\text{eff}}(G^{\wedge q+1}) \).
Proposition 7. Suppose \( A \) is a closed subscheme of codimension \( c \). Let \( X = X - Z \). There is an exact triangle of graded \( H(X) \)-modules.

\[
\begin{array}{ccc}
H(Z)[2c, c] & \xrightarrow{a} & H(X) \\
\downarrow & & \downarrow \\
H(U) & \xrightarrow{H(\text{id})} & H(X)
\end{array}
\]
Proof. For the most part, this is straightforward. The stating point is the exact triangle

\[
M(U) \rightarrow M(X) \rightarrow M(Z)(c)[2c] \rightarrow M(U)[1]
\]

Since $D^b_{\text{Nis}}$ is a tensor triangulated category, there is an exact triangle

\[
M(X) \otimes M(U) \rightarrow M(X) \otimes M(X) \rightarrow M(X) \otimes M(Z)(c)[2c] \rightarrow M(X) \otimes M(U)[1]
\]

The commutative diagram of schemes

\[
\begin{array}{ccc}
X \times U & \rightarrow & X \times X \\
\downarrow & & \downarrow \\
U \times U & \rightarrow & X
\end{array}
\]

\[
\begin{array}{ccc}
\Delta & & \\
\downarrow & & \\
U & \rightarrow & X
\end{array}
\]

gives us the following map of exact triangles

\[
\begin{array}{ccc}
M(X) \otimes M(U) & \rightarrow & M(X) \otimes M(X) \\
\downarrow & & \downarrow \\
M(U) & \rightarrow & M(X) \\
\downarrow & & \downarrow \\
M(U) & \rightarrow & M(Z)(c)[2c] \\
\downarrow & & \downarrow \\
M(U) & \rightarrow & M(U)[1]
\end{array}
\]

The dashed arrow exists because of the abstract nonsense of triangulated categories, see for instance [Wei94].

There is a pre-ordained map $M(Z) \rightarrow M(X) \otimes M(Z)$ given by the map $Z \rightarrow X$, so there is a map $M(Z) \otimes Z(c)[2c] \rightarrow M(X) \otimes M(Z) \otimes Z(c)[2c]$. We prove this map gives $H(Z)$ the same left $H(X)$-module structure as that in the gysin sequence.

Let $X'$ denote the blow-up of $X$ at $Z$, with $Z'$ the exceptional divisor, and $X''$ the blow up of $X \times \mathbb{A}^1$ at $Z \times \{0\}$, with $Z''$ the exceptional divisor. There is a cartesian cube

\[
\begin{array}{ccc}
Z'' & \rightarrow & X'' \\
\downarrow & & \downarrow \\
Z' & \rightarrow & X' \\
\downarrow & & \downarrow \\
Z \times \{0\} & \rightarrow & X \times \mathbb{A}^1 \\
\downarrow & & \downarrow \\
Z & \rightarrow & X
\end{array}
\]

From this it follows all spaces concerned are spaces over $X$.

The gysin map $M(X) \rightarrow M(Z)(c)[2c]$ is constructed in [MVW06][chapter 15] as a composite

\[
M(X) \rightarrow M(Z) \oplus M(X') \rightarrow M(Z \times \{0\}) \oplus M(X'') \rightarrow M(Z'') \rightarrow M(Z)(c)[2c]
\]

One can apply $M(X) \otimes \cdot$ throughout, and so the map $H(Z)[2c](c) \rightarrow H(X)$ is a left $H(X)$-module map, where $H(Z)$ is given the $H(X)$-module structure consistent with the following diagram

\[
\begin{array}{ccc}
M(X) \otimes M(Z') & \rightarrow & M(X) \otimes M(Z'') \\
\downarrow & & \downarrow \\
M(Z) \otimes M(Z') & \rightarrow & M(Z) \otimes M(Z'') \\
\downarrow & & \downarrow \\
M(Z') & \rightarrow & M(Z'') \\
\downarrow & & \downarrow \\
M(Z) & \rightarrow & M(Z) \otimes Z[2c](c)
\end{array}
\]

Since $Z'$ and $Z''$ are projectivizations of vector bundles $N$ and $N \oplus O_Z$ respectively, the $H(Z)$-module structure on $H(Z)[2c](c)$ is the usual one, as given by [Voe03], and since the $H(X)$-module structure factors through the $H(Z)$-module structure, the result follows. □
REFERENCES

