A LOCAL REGULARITY THEOREM FOR MEAN CURVATURE FLOW

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Abstract. This paper proves curvature bounds for mean curvature flows and other related flows in regions of spacetime where the Gaussian densities are close to 1.

Introduction

Let M_t with $0 < t < T$ be a smooth one-parameter family of embedded manifolds, not necessarily compact, moving by mean curvature in $\mathbf{R}^N$. This paper proves uniform curvature bounds in regions of spacetime where the Gaussian density ratios are close to 1. For instance (see §3.4):

Theorem. *There are numbers $\epsilon = \epsilon(N) > 0$ and $C = C(N) < \infty$ with the following property. If $\mathcal{M}$ is a smooth, proper mean curvature flow in an open subset U of the spacetime $\mathbf{R}^N \times \mathbf{R}$ and if the Gaussian density ratios of $\mathcal{M}$ are bounded above by $1+\epsilon$, then at each spacetime point $X = (x,t)$ of $\mathcal{M}$, the norm of the second fundamental form of $\mathcal{M}$ at X is bounded by*

$$\frac{C}{\delta(X,U)}$$

where $\delta(X,U)$ is the infimum of $\|X - Y\|$ among all points $Y = (y,s) \in U^c$ with $s \le t$.

(The terminology will be explained in section 2.)

Another paper [W5] extends the bounds to arbitrary mean curvature flows of integral varifolds. However, that extension seems to require Brakke's Local Regularity Theorem [B 6.11], the proof of which is very difficult. The results of this paper are much easier to prove, but nevertheless suffice in many interesting situations. In particular:

(1) The theory developed here applies up to and including the time at which singularities first occur in any classical mean curvature flow. (See Theorem 3.5.)

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(2) The bounds carry over easily to any varifold flow that is a weak limit of smooth mean curvature flows. (See §7.) In particular, any smooth compact embedded hypersurface of $\mathbf{R}^N$ is the initial surface of such a flow for $0 \le t < \infty$ (§7.4).

(3) The bounds also extend easily to any varifold flow constructed by Ilmanen's elliptic regularization procedure [I1].

Thus, for example, the results of this paper allow one to prove (without using Brakke's Local Regularity Theorem) that for a non-fattening mean curvature flow in $\mathbf{R}^N$, the surface is almost everywhere regular at all but countably many times. (A slightly weaker partial regularity result was proved using Brakke's Theorem by Evans and Spruck [ES4] and by Ilmanen [I1].) Similarly, the local regularity theorem here suffices (in place of Brakke's) for the analysis in [W 3,4] of mean curvature flow of mean convex hypersurfaces (see §7.2, §7.3, and §7.4).

The proofs here are quite elementary. They are based on nothing more than the Schauder estimates for the standard heat equation in $\mathbf{R}^m$ (see §8.2 and §8.3), and the fact that a non-moving plane is the only entire mean curvature flow with Gaussian density ratios everwhere equal to 1. The proof of the basic theorem is also fairly short; most of this paper is devoted to generalizations and extensions.

Although the key idea in the proof of the main theorem is simple, there are a number of technicalities in the execution. It may therefore be helpful to the reader to first see a simpler proof of an analogous result in which the key idea appears but without the technicalities. Such a proof (of a special case of Allard's Regularity Theorem) is given in section 1.

Section 2 contains preliminary definitions and lemmas. The main result of the paper is proved in section 3. In section 4, the result is extended to surfaces moving with normal velocity equal to mean curvature plus any Hölder continuous function of position, time, and tangent plane direction. This includes, for example, mean curvature flow in Riemannian manifolds (regarded as isometrically embedded in Euclidean space). In section 5, the analogous estimates at the boundary (or "edge") are proved for motion of manifolds-with-boundary. In section 6, somewhat weaker estimates (namely $C^{1,\alpha}$ and $W^{2,p}$) are proved for surfaces moving by mean curvature plus a bounded measurable function. This includes, for example, motion by mean curvature in the presence of smooth obstacles.

Finally, in section 7, the regularity theory is extended to certain mean curvature flows of varifolds. This section may be read directly after section 3, but it has been placed at the end of the paper since it is the only section involving varifolds.

Mean curvature flow has been investigated extensively in the last few decades. Three distinct approaches have been very fruitful in those investigations: geometric measure theory, classical PDE, and the theory of level-set or viscosity solutions. These were pioneered in [B]; [H1] and [GH]; and [ES 1–4] and [CGG] (see also [OS]), respectively. Surveys emphasizing the classical PDE approach may be found in [E2] and [H3]. A rather thorough introduction to the classical approach, including some new results, may be found in [E4]. An introduction to the geometric measure theory and viscosity approaches is included in [I1]. See [G] for a more extensive introduction to the level set approach.

Some of the results in this paper were announced in [W1]. Some similar results were proved by A. Stone [St1,2] for the special case of hypersurfaces with positive mean curvature under an additional hypothesis about the rate at which curvature blows up when singularities first appear. In [E1], K. Ecker proved, for the special case of two-dimensional surfaces in 3-manifolds, pointwise curvature bounds assuming certain integral curvature bounds. The monotonicity formula, which plays a major role in this paper, was discovered by G. Huisken [H2]. Ecker has recently discovered two new remarkable monotonicity formulas [E3, E4 §3.18] that have most of the desirable features of Huisken's (and that yield the same infinitesimal densities) but that, unlike Huisken's, are local in space.

1. The Main Idea of the Proof

As mentioned above, the key idea in the proof of the main theorem is simple, but there are a number of necessary technicalities that obscure the idea. In this section, the same idea (minus the technicalities) is used to prove a special case of Allard's Regularity Theorem [A]. The proof is followed by a brief discussion of the some of the technicalities that make the rest of the paper more complicated.

This section is included purely for expository reasons and may be skipped.

1.1. Theorem. *Suppose that N is a compact Riemannian manifold and that $\rho > 0$. There exist positive numbers $\epsilon = \epsilon(N,\rho)$ and $C = C(N,\rho)$ with the following property. If M is a smooth embedded minimal submanifold of N such that*

$$\theta(M,x,r) \le 1+\epsilon$$

for all $x \in N$ and $r \le \rho$, then the norm of the second fundamental form of M is everywhere bounded by C.

Here $\theta(M,x,r)$ denotes the density ratio of M in $\mathbf{B}(x,r)$:

$$\theta(M,x,r) = \frac{\operatorname{area}(M \cap \mathbf{B}(x,r))}{\omega_m r^m},$$

where $m = \dim(M)$ and ω_m is the volume of the unit ball in $\mathbf{R}^m$.

Proof. Suppose the result were false for some N and ρ. Then there would be a sequence $\epsilon_i \to 0$ of positive numbers, a sequence M_i of smooth minimal submanifolds of N, and a sequence x_i of points in M_i for which

$$(*) \qquad \theta(M_i,x,r) \le 1+\epsilon_i \quad (x \in N, r \le \rho)$$

and for which

$$B(M_i,x_i) \to \infty,$$

where $B(M,x)$ denotes the norm of the second fundamental form of M at x. Note that we may choose the x_i to maximize $B(M_i,\cdot)$:

$$\max_x B(M_i,x) = B(M_i,x_i) = \Lambda_i \to \infty.$$

We may also assume that N is isometrically embedded in a Euclidean space $\mathbf{E}$. Translate M_i by $-x_i$ and dilate by Λ_i to get a new manifold M_i' with

$$\max B(M_i',\cdot) = B(M_i',0) = 1.$$

By an Arzela-Ascoli argument, a subsequence (which we may assume is the original sequence) of the M_i' converges in $C^{1,\alpha}$ to a limit submanifold M' of $\mathbf{E}$. By standard elliptic PDE, the convergence is in fact smooth, so that M' is a minimal submanifold of $\mathbf{E}$ and

$$(\dagger) \qquad \max B(M', \cdot) = B(M', 0) = 1.$$

On the other hand, (*) implies that

$$\theta(M', x, r) \leq 1 \text{ for all } x \in M' \text{ and for all } r.$$

Monotonicity implies that the only minimal surface with this property is a plane. So M' is a plane. But that contradicts (†). □

Complications

There are several reasons why the proof of the main theorem (3.1) of this paper is more complicated than the proof above. For example:

1. It is much more useful to have a local result than a global one. Thus in Theorem 1.1, it would be better to assume not that M is compact, but rather that it is a proper submanifold of an open subset U of N. Of course then we can no longer conclude that $B(M, \cdot)$ is bounded. Instead, the assertion should become

$$B(M, x)\,\mathrm{dist}(x, U^c) \leq 1.$$

This localization introduces a few annoyances into the proof. For example, we would like (following the proof above) to choose a point $x_i \in M_i$ for which

$$B(M_i, x_i)\,\mathrm{dist}(x_i, U_i^c)$$

is a maximum. But it is not clear that this quantity is even bounded, and even if it is bounded, the supremum may not be attained.

2. For various reasons, it is desirable to have a slightly more complicated quantity play the role that $B(M, x)$ does above. For instance, $\max B(M, \cdot)$ is like the C^2 norm of a function, and as is well known, Schauder norms are much better suited to elliptic and parabolic PDE's. Thus instead of $B(M, x)$ we use a quantity $K_{2,\alpha}(M, x)$ which is essentially the smallest number $\lambda > 0$ such that the result of dilating $M \cap \mathbf{B}(x, 1/\lambda)$ by λ is, after a suitable rotation, contained in the graph of a function

$$u : \mathbf{R}^m \to \mathbf{R}^{d-m}$$

with $\|u\|_{2,\alpha} \leq 1$. Here d is the dimension of the ambient Euclidean space.

There is another reason for not using the norm of the second fundamantal form. Suppose we wish to weaken the hypothesis of Theorem 1.1 by requiring not that M be minimal but rather that its mean curvature be bounded. We can then no longer conclude anything about curvatures. However, we can still conclude, with essentially the same proof, that $K_{1,\alpha}(M, x)$ is bounded.

3. Spacetime (for parabolic problems) is somewhat more complicated than space (for elliptic problems). Thus for example parabolic dilations and Gaussian densities replace the more geometrically intuitive Euclidean dilations and densities.

2. Preliminaries

2.1. Spacetime. We will work in spacetime $\mathbf{R}^{N,1} = \mathbf{R}^N \times \mathbf{R}$. Points of spacetime will be denoted by capital letters: X, Y, etc. If $X = (x,t)$ is a point in spacetime, $\|X\|$ denotes its parabolic norm:

$$\|X\| = \|(x,t)\| = \max\{|x|, |t|^{1/2}\}.$$

The norm makes spacetime into a metric space, the distance between X and Y being $\|X - Y\|$. Note that the distance is invariant under spacetime translations and under orthogonal motions of $\mathbf{R}^N$.

For $\lambda > 0$, we let $\mathcal{D}_\lambda : \mathbf{R}^{N,1} \to \mathbf{R}^{N,1}$ denote the parabolic dilation:

$$\mathcal{D}_\lambda(x,t) = (\lambda x, \lambda^2 t).$$

Note that $\|\mathcal{D}_\lambda X\| = \lambda \|X\|$.

We let $\tau : \mathbf{R}^{N,1} \to \mathbf{R}$ denote projection onto the time axis:

$$\tau(x,t) = t.$$

2.2. Regular flows. Let $\mathcal{M}$ be a subset of $\mathbf{R}^{N,1}$ such that $\mathcal{M}$ is a C^1 embedded submanifold (in the ordinary Euclidean sense) of $\mathbf{R}^{N+1}$ with dimension $m+1$ (again, in the usual Euclidean sense). If the time function $\tau : (x,t) \mapsto t$ has no critical points in $\mathcal{M}$, then we say that $\mathcal{M}$ is a **fully regular flow** of spatial dimension m. If a fully regular flow $\mathcal{M}$ is C^∞ as a submanifold of $\mathbf{R}^{N+1}$, then we say that it is a **fully smooth flow**.

It is sometimes convenient to allow flows that softly and suddenly vanish away. Thus if $\mathcal{M}$ is a fully regular (or fully smooth) flow and $T \in (-\infty, \infty]$, then the truncated set

$$\{X \in \mathcal{M} : \tau(X) \le T\}$$

will be called a **regular** (or **smooth**) flow. Of course if $T = \infty$, then the truncation has no effect. Thus every fully regular (or fully smooth) flow is also a regular (or smooth) flow.

Note that if $\mathcal{M}$ is a regular (or smooth) flow, then for each $t \in \mathbf{R}$, the spatial slice

$$\mathcal{M}(t) := \{x \in \mathbf{R}^N : (x,t) \in \mathcal{M}\}$$

is a C^1 (or smooth) m-dimensional submanifold of $\mathbf{R}^N$. Of course for some times t the slice may be empty.

For example, suppose M is a smooth m-dimensional manifold, I is an interval of the form (a,b) or $(a,b]$, and

$$F : M \times I \to \mathbf{R}^N$$

is a smooth map such that for each $t \in I$, the map $F(\,\cdot\,,t) : M \to \mathbf{R}^N$ is an embedding. Let $\mathcal{M}$ be the set in spacetime traced out by F:

$$\mathcal{M} = \{(F(x,t),t) : x \in M, \quad t \in I\}.$$

Then $\mathcal{M}$ is a smooth flow.

Conversely, if $\mathcal{M}$ is any regular (or smooth) flow and $X \in \mathcal{M}$, then there is a spacetime neighborhood U of X and an F as above such that

$$\mathcal{M} \cap U = \{(F(x,t),t) : x \in M, \quad t \in I\}$$

is the flow traced out by F. Such an F is called a **local parametrization** of $\mathcal{M}$. If $\mathcal{M}$ is smooth, then by the Fundamental Existence and Uniqueness Theorem for ODEs, we can choose F so that for all (x,t) in the domain of F, the time derivative $\frac{\partial}{\partial t}F(x,t)$ is perpendicular to $F(M,t)$ at $F(x,t)$.

2.3. Proper flows. Suppose that $\mathcal{M}$ is a regular flow and that U is an open subset of spacetime. If

$$\mathcal{M} = \overline{\mathcal{M}} \cap U,$$

then we will say that $\mathcal{M}$ is a **proper flow** in U.

For any regular flow $\mathcal{M}$, if U is the spacetime complement of $\overline{\mathcal{M}} \setminus \mathcal{M}$, then $\mathcal{M}$ is a proper flow in U. Also, if $\mathcal{M}$ is a proper flow in U and if U' is an open subset of U, then $\mathcal{M} \cap U'$ is a proper flow in U'.

2.4. Normal velocity and mean curvature. Let $\mathcal{M}$ be a regular flow in $\mathbf{R}^{N,1}$. Then for each $X = (x,t) \in \mathcal{M}$, there is a unique vector $v = \mathrm{v}(\mathcal{M}, X)$ in $\mathbf{R}^N$ such that v is normal to $\mathcal{M}(t)$ at x and $(v,1)$ is tangent (in the ordinary Euclidean sense) to $\mathcal{M}$ at X. This vector is called the **normal velocity** of $\mathcal{M}$ at X. If F is a local parametrization of $\mathcal{M}$, then

$$\mathrm{v}(\mathcal{M}, (F(x,t),t)) = \left(\frac{\partial}{\partial t} F(x,t) \right)^{\perp}.$$

If $\mathcal{M}$ is a regular flow and $X = (x,t)$, we let $\mathrm{H}(\mathcal{M}, X)$ denote the mean curvature vector (if it exists) of $\mathcal{M}(t)$ at x. Of course if $\mathcal{M}$ is smooth, then $\mathcal{M}(t)$ is also smooth, so $\mathrm{H}(\mathcal{M}, X)$ does exist. A regular flow $\mathcal{M}$ such that $\mathrm{v}(\mathcal{M}, X) = \mathrm{H}(\mathcal{M}, X)$ for all $X \in \mathcal{M}$ is called a **mean curvature flow**.

Note that if we parabolically dilate $\mathcal{M}$ by λ, then v and H get multiplied by $1/\lambda$:

$$\mathrm{v}(\mathcal{D}_\lambda \mathcal{M}, \mathcal{D}_\lambda X) = \lambda^{-1}\, \mathrm{v}(\mathcal{M}, X),$$
$$\mathrm{H}(\mathcal{D}_\lambda \mathcal{M}, \mathcal{D}_\lambda X) = \lambda^{-1}\, \mathrm{H}(\mathcal{M}, X).$$

Thus if $\mathcal{M}$ is a mean curvature flow, then so is $\mathcal{D}_\lambda \mathcal{M}$.

2.5. The $C^{2,\alpha}$ norm of $\mathcal{M}$ at X.

We wish to define a kind of local $C^{2,\alpha}$ norm of a smooth flow at a point $X \in \mathcal{M}$. This norm will be denoted $K_{2,\alpha}(\mathcal{M}, X)$. Actually the definition below makes sense for any subset $\mathcal{M}$ of spacetime. Let $\mathbf{B}^N = \mathbf{B}^N(0,1)$ and $\mathbf{B}^{N,1} = \mathbf{B}^N \times (-1,1)$ denote the open unit balls centered at the origin in $\mathbf{R}^N$ and in spacetime $\mathbf{R}^{N,1}$, respectively. The *graph* of a function $u : \mathbf{B}^{m,1} \to \mathbf{R}^{N-m}$ is the set

$$\mathrm{graph}(u) = \{(x, u(x,t), t) : (x,t) \in \mathbf{B}^{m,1}\} \subset \mathbf{R}^{N,1}.$$

Now consider first the case $X = 0 \in \mathcal{M}$. Suppose we can rotate $\mathcal{M}$ to get a new set $\mathcal{M}'$ for which the intersection

$$\mathcal{M}' \cap \mathbf{B}^{N,1}$$

is contained in the graph of a function

$$u : \mathbf{B}^{m,1} \to \mathbf{R}^{N-m}$$

whose parabolic $C^{2,\alpha}$ norm is ≤ 1. (See the appendix (§7) for the definition of the parabolic Hölder norms of functions.) Then we will say that

$$K_{2,\alpha}(\mathcal{M}, X) = K_{2,\alpha}(\mathcal{M}, 0) \le 1.$$

Otherwise, $K_{2,\alpha}(\mathcal{M}, 0) > 1$.

More generally, we let

$$K_{2,\alpha}(\mathcal{M}, 0) = \inf\{\lambda > 0 : K_{2,\alpha}(\mathcal{D}_\lambda \mathcal{M}, 0) \le 1\}.$$

Finally, if X is any point in $\mathcal{M}$, we let

$$K_{2,\alpha}(\mathcal{M}, X) = K_{2,\alpha}(\mathcal{M} - X, 0),$$

where $\mathcal{M} - X$ is the flow obtained from $\mathcal{M}$ by translating in spacetime by $-X$.

If $K_{2,\alpha}(\mathcal{M}, \cdot)$ is bounded on compact subsets of a regular flow $\mathcal{M}$, then $\mathcal{M}$ is called a $C^{2,\alpha}$ flow.

Remark on the definition.. Suppose $\mathcal{M}$ is a proper, regular flow in U and $X \in \mathcal{M}$. If we translate $\mathcal{M}$ by $-X$, dilate by $\lambda = K_{2,\alpha}(\mathcal{M}, X)$, and then rotate appropriately to get a flow $\mathcal{M}'$, then by definition,

$$\mathcal{M}' \cap \mathbf{B}^{N,1}$$

will be contained in the graph of a function

$$u : \mathbf{B}^{m,1} \to \mathbf{R}^{N-m}$$

as with parabolic $C^{2,\alpha}$ norm ≤ 1. Note that if $\mathcal{M}$ is fully regular and if the distance from X to U^c is $\ge r = 1/\lambda$, then in fact

$$\mathcal{M}' \cap \mathbf{B}^{N,1} = \text{graph}(u) \cap \mathbf{B}^{N,1}.$$

Likewise, if $\mathcal{M}$ is regular but not necessarily fully regular, then for some $T \ge 0$,

$$\mathcal{M}' \cap \mathbf{B}^{N,1} = \text{graph(u)} \cap \mathbf{B}^{N,1} \cap \{Y : \tau(Y) \le T\}.$$

2.6. Arzela-Ascoli Theorem. *For $i = 1, 2, 3, \ldots$, let $\mathcal{M}_i$ be a proper $C^{2,\alpha}$ flow in U_i. Suppose that $\mathcal{M}_i \to \mathcal{M}$ and that $U_i^c \to U^c$ as sets. Suppose also that the functions $K_{2,\alpha}(\mathcal{M}_i, \cdot)$ are uniformly bounded as $i \to \infty$ on compact subsets of U. Then $\mathcal{M}' = \mathcal{M} \cap U$ is a proper $C^{2,\alpha}$ flow in U, and the convergence $\mathcal{M}_i \to \mathcal{M}'$ is locally C^2 (parabolically). In particular, if $X_i \in \mathcal{M}_i$ converges to $X \in \mathcal{M}'$, then*

$$\mathrm{v}(\mathcal{M}_i, X_i) \to \mathrm{v}(\mathcal{M}', X),$$
$$\mathrm{H}(\mathcal{M}_i, X_i) \to \mathrm{H}(\mathcal{M}', X),$$

and

$$K_{2,\alpha}(\mathcal{M}', X) \le \liminf_i K_{2,\alpha}(\mathcal{M}_i, X_i).$$

Furthermore, if each $\mathcal{M}_i$ is fully regular in U_i, then $\mathcal{M}$ is fully regular in U.

Remark on the hypotheses:. Convergence of $S_i \to S$ as sets means: every point in S is the limit of a sequence $X_i \in S_i$, and for every bounded sequence $X_i \in S_i$, all subsequential limits lie in S. The uniform boundedness hypothesis is equivalent to: for every sequence $X_i \in \mathcal{M}_i$ converging to $X \in U$, the limsup of $K_{2,\alpha}(\mathcal{M}_i, X_i)$ is finite.

Proof. Straightforward. See §8.1 for details. The last assertion follows from the remark above about the definition of $K_{2,\alpha}$.

Note that $K_{2,\alpha}(\mathcal{M}, \cdot)$ scales like the reciprocal of distance. That is,

$$K_{2,\alpha}(\mathcal{D}_\lambda \mathcal{M}, \mathcal{D}_\lambda X) = \lambda^{-1} K_{2,\alpha}(\mathcal{M}, X).$$

We will also need a scale invariant version of $K_{2,\alpha}$. Let

$$d(X, U) = \inf\{\|X - Y\| : Y \in U^c\}.$$

Then of course $d(X, U)\, K_{2,\alpha}(\mathcal{M}, X)$ is scale invariant.

Definition. Suppose $\mathcal{M}$ is a proper smooth flow in U. Then

$$K_{2,\alpha;\,U}(\mathcal{M}) = \sup_{X \in \mathcal{M}} d(X, U) \cdot K_{2,\alpha}(\mathcal{M}, X).$$

Of course $K_{2,\alpha;\,U}(\mathcal{M})$ is scale-invariant.

2.7. Corollary to the Arzela-Ascoli Theorem.

$$K_{2,\alpha;\,U}(\mathcal{M}') \le \liminf K_{2,\alpha;\,U_i}(\mathcal{M}_i)$$

2.8. Proposition. *Suppose $\mathcal{M}$ is a proper $C^{2,\alpha}$ flow in U. Let $U_1 \subset U_2 \subset \dots$ be open sets such that*

(1) *the closure of each U_i is a compact subset of U, and*
(2) $\cup_i U_i = U$.

Then

$$K_{2,\alpha;\,U_i}(\mathcal{M} \cap U_i) < \infty$$

for each i and

$$K_{2,\alpha;U}(\mathcal{M}) = \lim K_{2,\alpha;\,U_i}(\mathcal{M} \cap U_i).$$

The proof is very easy. Note that for any U, there always exist such U_i. For instance, we can let $U_i = \{X \in U : d(X, U) > 1/i \text{ and } \|X\| < i\}$.

2.9. Gaussian density. If $\mathcal{M}$ is a regular flow in $\mathbf{R}^{N,1}$ with spatial dimension m, if $X \in \mathbf{R}^{N,1}$, and if $r > 0$, then the **Gaussian density ratio** of $\mathcal{M}$ at X with radius r is

$$\Theta(\mathcal{M}, X, r) = \int_{y \in \mathcal{M}(t-r^2)} \frac{1}{(4\pi r^2)^{m/2}} \exp\left(\frac{-|y-x|^2}{4r^2}\right) \, d\mathcal{H}^m y.$$

If $\mathcal{M}$ is a proper mean curvature flow in $\mathbf{R}^N \times (a,b)$ and if $\tau(X) > a$, then the density ration $\Theta(\mathcal{M}, X, r)$ is a non-decreasing function of r for $0 < r < \sqrt{\tau(X) - a}$. Thus the limit

$$\Theta(\mathcal{M}, X) := \lim_{r \to 0} \Theta(\mathcal{M}, X, r)$$

exists for all X, and is called the **Gaussian density** of $\mathcal{M}$ at X. Similarly, if $\mathcal{M}$ is proper in all of $\mathbf{R}^{N,1}$, then the limit

$$\Theta(\mathcal{M}, \infty) := \lim_{r \to \infty} \Theta(\mathcal{M}, X, r)$$

exists for all X. It is easy to show that, as the notation indicates, the limit is independent of X.

Analogous, slightly weaker statements are true for related flows (as in sections 4, 5, and 6 of this paper), as well as for proper mean curvature flows in more general open subsets U of spacetime: see sections 10 and 11 of [W2]. (The notations here and in [W2] differ slightly. The quantity denoted here by $\Theta(\mathcal{M}, X, r)$ is written there as $\Theta(\mathcal{M}, X, \tau)$ where $\tau = r^2$. The notation here makes more apparent the analogy between mean curvature flows and minimal surfaces.)

The proof of the Monotonicity Theorem (see [H2] or [I2]) shows that if $\mathcal{M}$ is a proper mean curvature flow in $\mathbf{R}^{N,1}$ and if

$$\Theta(\mathcal{M}, X) = \Theta(\mathcal{M}, \infty),$$

then the flow

$$\mathcal{M}' = (\mathcal{M} - X) \cap \{Y : \tau(Y) \leq 0\}$$

is invariant under parabolic dilations:

$$\mathcal{M}' \equiv \mathcal{D}_\lambda \mathcal{M}'.$$

Taking the limit of this equation as $\lambda \to \infty$ shows that if $\mathcal{M}'$ is smooth at X, then $\mathcal{M}'$ has the form (after a suitable rotation) $\mathbf{R}^m \times [0]^{N-k} \times (-\infty, 0]$.

2.10. Proposition. *Suppose $\mathcal{M}$ is a smooth non-empty proper mean-curvature flow in $\mathbf{R}^{N,1}$. Then $\Theta(\mathcal{M}, \infty) \geq 1$, with equality if and only if $\mathcal{M}$ has the form*

$$H \times (-\infty, T] \tag{*}$$

for some affine plane $H \subset \mathbf{R}^N$ and some $T \in (-\infty, \infty]$.

Proof. Let $X \in \mathcal{M}$. We claim that $\Theta(\mathcal{M}, X) \geq 1$. (In fact equality holds, but we do not need that here.) The claim may be proved as follows. Note that the dilates

$\mathcal{D}_\lambda(\mathcal{M}-X)$ converge, as $\lambda\to\infty$, to a limit flow $\mathcal{M}'$ of the form (*). (Here $\mathcal{M}-X$ denotes the result of translating $\mathcal{M}$ in spacetime by $-X$.) Hence

$$\begin{aligned}\Theta(\mathcal{M}',0,1) &\le \lim_{\lambda\to\infty}\Theta(\mathrm{D}_\lambda(\mathcal{M}-X),0,r)\\ &\le \lim_{\lambda\to\infty}\Theta(\mathcal{M}-X,0,r/\lambda)\\ &= \lim_{\lambda\to\infty}\Theta(\mathcal{M},X,r/\lambda)\\ &= \Theta(\mathcal{M},X).\end{aligned}$$

But $\Theta(\mathcal{M}',0,1)=1$ by direct calculation. Hence $\Theta(\mathcal{M},X)\ge 1$ for each $X\in\mathcal{M}$. Thus by monotonicity, $\Theta(\mathcal{M},\infty)\ge 1$. Furthermore, if $\Theta(\mathcal{M},\infty)\le 1$, then

$$\Theta(\mathcal{M},\infty)=\Theta(\mathcal{M},X)=1$$

for every $X\in\mathcal{M}$. But by monotonicity (see the discussion immediately preceding 2.10), this implies that the set

$$\{Y\in\mathcal{M}:\tau(Y)\le\tau(X)\}$$

has the form (*). Since this is true for every $X\in\mathcal{M}$, in fact all of $\mathcal{M}$ must have the form (*). □

3. The Fundamental Theorem

3.1. Theorem. *For $0<\alpha<1$, there exist positive numbers $\epsilon=\epsilon(N,m,\alpha)$ and $C=C(N,m,\alpha)$ with the following property. Suppose $\mathcal{M}$ is a smooth proper mean curvature flow with spatial dimension m in $U\subset\mathbf{R}^{N,1}$ such that*

$$\Theta(\mathcal{M},X,r)\le 1+\epsilon \textit{ for all } X\in\mathcal{M} \textit{ and } 0<r<d(X,U).$$

Then

$$K_{2,\alpha;\,U}(\mathcal{M})\le C.$$

Remark. Of course the bound on $K_{2,\alpha;\,U}(\mathcal{M})$ immediately implies, by standard classical PDE, bounds on all higher derivatives.

Proof. Let $\bar\epsilon$ be the infimum of numbers $\epsilon>0$ for which the theorem fails, i.e., for which there is no appropriate $C<\infty$. We must show that $\bar\epsilon>0$.

Let $\epsilon_i>\bar\epsilon$ be a sequence of numbers converging to $\bar\epsilon$. Then there are sequences $\mathcal{M}_i$ and U_i such that $\mathcal{M}_i$ is a smooth proper mean curvature flow in U_i and such that

$$\Theta(\mathcal{M}_i,X,r)\le 1+\epsilon_i \quad \text{for all } X\in\mathcal{M}_i \text{ and } 0<r<d(X,U_i), \tag{1}$$

$$K_{2,\alpha;U_i}(\mathcal{M}_i)\to\infty. \tag{2}$$

By Proposition 2.8, we can assume that

$$K_{2,\alpha;U_i}(\mathcal{M}_i)=s_i<\infty$$

for each i. (Otherwise replace $\mathcal{M}_i$ and U_i by $\mathcal{M}_i'$ and U_i', where U_i' is compactly contained in U_i and $\mathcal{M}_i' = \mathcal{M}_i \cap U_i'$. Hypothesis (1) will still hold, and by Proposition 2.8, we can choose the U_i' large enough that (2) will still hold.)

Choose $X_i \in \mathcal{M}_i$ so that

$$d(X_i, U_i)\, K_{2,\alpha}(\mathcal{M}_i, X_i) > \frac{1}{2} s_i. \tag{3}$$

By translating, we may assume that $X_i \equiv 0$. By dilating, we may assume that

$$K_{2,\alpha}(\mathcal{M}_i, 0) = 1. \tag{4}$$

Of course by (2) and (3) this means

$$d(0, U_i) \to \infty. \tag{5}$$

Now let $X \in \mathcal{M}_i$. Then

$$\begin{aligned} d(X, U_i) K_{2,\alpha}(\mathcal{M}_i, X) &\le s_i \\ &\le 2d(0, U_i) K_{2,\alpha}(\mathcal{M}_i, 0) \\ &= 2d(0, U_i). \end{aligned}$$

Thus

$$\begin{aligned} K_{2,\alpha}(\mathcal{M}_i, X) &\le 2\frac{d(0, U_i)}{d(X, U_i)} \\ &\le 2\frac{d(0, U_i)}{d(0, U_i) - \|X\|} \\ &= 2\left(1 - \frac{\|X\|}{d(0, U_i)}\right)^{-1} \end{aligned}$$

provided the right hand side is positive. By (5), this means that $K_{2,\alpha}(\mathcal{M}_i, \cdot)$ is uniformly bounded as $i \to \infty$ on compact subset of spacetime.

Thus by the Arzela-Ascoli Theorem 2.6, a subsequence (which we may assume is the original sequence) of the $\mathcal{M}_i$ converges locally to a limit mean curvature flow $\mathcal{M}$ that is proper in all of $\mathbf{R}^{N,1}$ (because by (5), $U_i^c \to \emptyset$).

Note that

$$\Theta(\mathcal{M}, X, r) \le 1 + \bar{\epsilon} \tag{6}$$

for all X and $r > 0$.

Now suppose that $\bar{\epsilon} = 0$; we will show that this leads to a contradiction.

By monotonicity (see §2.10), the inequality (6) (with $\bar{\epsilon} = 0$) implies that $\mathcal{M}$ has the form (after a suitable rotation):

$$\mathcal{M} = \mathbf{R}^m \times [0]^{N-m} \times (-\infty, T]$$

for some $T \in [0, \infty]$. Here $T = \lim_i T_i$ where $T_i = \sup\{\tau(X) : X \in \mathcal{M}_i\}$. If $T = \infty$, then $(-\infty, T]$ should be interpreted as $\mathbf{R}$.

By the C^2 convergence, there exist $r_i \to \infty$ such that

$$\mathcal{M}_i \cap (\mathbf{B}^m(r_i) \times \mathbf{B}^{N-m}(r_i) \times (-r_i, r_i))$$

is the graph of a function

$$u_i : \mathbf{B}^m(r_i) \times I_i \to \mathbf{R}^{N-m}$$

with

$$\|u_i\|_{C^2} \to 0.$$

Here I_i is the interval $(-r_i, r_i) \cap (-\infty, T_i]$.

Of course the u_i satisfy the non-parametric mean curvature flow equation:

$$\frac{\partial}{\partial t} u_i - \Delta u_i = f_i, \tag{7}$$

where

$$f_i = - \sum_{1 \le j,k \le m} \frac{\mathrm{D}_j u_i \, \mathrm{D}_k u_i}{1 + |\,\mathrm{D} u_i|^2} \, \mathrm{D}_{jk} u_i. \tag{8}$$

(This is the equation for hypersurfaces. When $N > m+1$, the equation is more complicated (see the appendix), but the proof below is still valid.)

Now the f_i converge to 0 in C^α on compact sets. This is seen as follows. Recall that the u_i are uniformly bounded in $C^{2,\alpha}$ on compact sets and converge to 0 in C^2 on compact sets. Thus

$$\frac{\mathrm{D}_j u_i \, \mathrm{D}_k u_i}{1 + |\,\mathrm{D} u|^2} \tag{8}$$

converges to 0 in C^1 (on compact sets) and

$$\mathrm{D}_{jk} u_i \tag{9}$$

is bounded in C^α (on compact sets). It follows that the product of (8) and (9) converges to 0 in C^α on compact sets.

Thus the Schauder estimates (§8.2) for the heat equation (7) imply that

$$\|u_i | K\|_{2,\alpha} \to 0$$

for every compact $K \subset \mathbf{R}^{m,1}$. But that contradicts the fact that $K_{2,\alpha}(\mathcal{M}_i, 0)$ was normalized to be 1. □

In the statement of Theorem 3.1, ϵ was allowed to depend on α. But in fact it can be chosen independently of α:

3.2. Proposition. *Let $\overline{\epsilon}$ be the infimum of $\epsilon > 0$ for which Theorem 3.1 fails. Then*

(1) *There are no proper smooth mean curvature flows $\mathcal{M}$ in $\mathbf{R}^{N,1}$ with*

$$1 < \Theta(\mathcal{M}, \infty) < 1 + \overline{\epsilon}.$$

(2) *There is such a flow with*

$$\Theta(\mathcal{M}, \infty) = 1 + \overline{\epsilon}.$$

(3) *theorem 3.1 fails for $\epsilon = \overline{\epsilon}$.*

Of course from (1) and (2), it is clear that $\overline{\epsilon}$ does not depend on α.

Proof. To prove (1), suppose $\mathcal{M}$ is a smooth proper mean curvature flow in $\mathbf{R}^{N,1}$ with

$$\Theta(\mathcal{M},\infty) < 1+\overline{\epsilon}.$$

Then

$$\Theta(\mathcal{M},\infty) < 1+\epsilon$$

for some $\epsilon < \overline{\epsilon}$. By monotonicity,

$$\Theta(\mathcal{M},X,r) < 1+\epsilon$$

for all X and r. Thus by Theorem 3.1,

$$K_{2,\alpha}(\mathcal{M},X) \le \frac{C}{d(\mathcal{M},\mathbf{R}^{N,1})}.$$

But $d(\mathcal{M},\mathbf{R}^{N,1}) = \infty$, so $K_{2,\alpha}(\mathcal{M},\cdot) \equiv 0$, which implies that $\Theta(\mathcal{M},X,r) \equiv 1$ for all $X \in \mathcal{M}$. This proves (1).

The proof of Theorem 3.1 established that there is a smooth proper mean curvature flow $\mathcal{M}$ with

$$1 < \Theta(\mathcal{M},\infty) \le 1+\overline{\epsilon}$$

(The contradiction in that proof only came from assuming that $\Theta(\mathcal{M},\infty) = 1$.) This together with (1) gives (2).

Finally, (3) follows from (2) because the $\mathcal{M}$ of (2) is a counterexample to Theorem 3.1 for $\epsilon = \overline{\epsilon}$. □

3.3. Remark. Let $\mathcal{M}$ be a proper mean curvature flow in U. Note that the bounds given by theorem 3.1 at a point X depend on the distance from X to U^c. Of course U^c may include points Y with $\tau(Y) > \tau(X)$. Thus, at first glance, it may seem that to use Theorem 3.1 to deduce curvature bounds at a certain time $t = \tau(X)$ requires knowledge about the flow at subsequent times. But this is not really the case, since we can apply the theorem to the flow

$$\mathcal{M}' = \{X \in \mathcal{M} : \tau(X) \le t\}$$

which is proper in the set

$$U' = U \cup \{Z : \tau(Z) > t\}$$

This gives bounds up to and including time t that do not depend on anything after time t. Because we can do this at each time t, we get the following corollary to theorem 3.1.

3.4. Corollary. *Let $\mathcal{M}$ be as in theorem 3.1. Then at every point X of $\mathcal{M}$, the norm of the second fundamental form of the spatial slice of $\mathcal{M}$ at X is bounded by*

$$\frac{C}{\delta(X,U)}$$

where $\delta(X,U)$ is the infimum of $\|X-Y\|$ among all points $Y \in U^c$ with $\tau(Y) \le \tau(X)$.

3.5. Theorem. *Let M be a compact m-dimensional manifold and let*

$$F : M \times [0, T) \to \mathbf{R}^N$$

be a classical mean-curvature flow. Let $\mathcal{M}$ be the subset of spacetime traced out by F during the time interval $(0, T)$. Suppose $X = (x, T)$ is a point such that $\Theta(\mathcal{M}, X) < 1 + \epsilon$. Then X is a regular point of $\overline{\mathcal{M}}$. That is, there is a spacetime neighborhood U of X such that

$$\overline{\mathcal{M}} \cap U$$

is a smooth flow.

Proof. By definition, there is an r with $0 < r < \sqrt{T}$ such that

$$\Theta(\mathcal{M}, X, r) < 1 + \epsilon.$$

It follows by continuity that

$$\Theta(\mathcal{M}, \cdot, r) < 1 + \epsilon$$

on some spacetime neighborhood U of X. Then by monotonicity,

$$\Theta(\mathcal{M}, Y, \rho) < 1 + \epsilon$$

for all $Y \in U$ and $\rho \le r$.

We may choose U small enough that its diameter is less than r.

Let $T_i < T$ be a sequence of times converging to T. Then the flows

$$\mathcal{M}_i := \{Y \in \mathcal{M} \cap U : \tau(Y) \le T_i\}$$

and the open set U satisfy the hypotheses of Theorem 3.1, so

$$K_{2,\alpha;\,U}(\mathcal{M}_i) \le C.$$

Passing to the limit gives

$$K_{2,\alpha;\,U}(\mathcal{M}') \le C,$$

where $\mathcal{M}' = \overline{\mathcal{M}} \cap U$. $\square$

3.6. Remark. Theorem 3.5 is almost, but not quite, a special case of Brakke's Local Regularity Theorem. To apply Brakke's theorem, we would need, for some $R > 0$, a certain positive lower bound on

$$\liminf_{t \to T} \frac{\text{area}(\mathcal{M}(t) \cap \mathbf{B}(x, R))}{R^m}.$$

Such a lower bound does not immediately follow from the hypotheses of 3.5.

4. Additional Forces

There are many interesting geometric evolutions closely related to mean curvature flow in Euclidean space. Consider for example:

(1) A compact embedded hypersurface in $\mathbf{R}^N$ moving by the gradient flow for the functional: area minus enclosed volume. Thus the velocity at each point will equal the mean curvature plus the outward pointing unit normal.
(2) mean curvature flow in the unit sphere $\mathbf{S}^{N-1}$.
(3) mean curvature flow in a compact Riemannian manifold S (which we may take to be isometrically embedded in $\mathbf{R}^N$.)

To handle such situations, it is convenient to introduce an operator $\beta(\mathcal{M})$ as follows. First, if $\mathcal{M}$ is a regular flow of m-dimensional surfaces in $\mathbf{R}^N$ and if $X = (x,t) \in \mathcal{M}$, we let $\mathrm{Tan}(\mathcal{M}, X)$ be the tangent plane (oriented or non-oriented as needed) to $\mathcal{M}(t)$ at x. Then we define the **Brakke operator**:

$$\beta(\mathcal{M}) : \text{a subset of } (\mathbf{R}^{N,1} \times G(m,N)) \to \mathbf{R}^N$$

by

$$\beta(X,V) = \mathrm{v}(\mathcal{M},X) - \mathrm{H}(\mathcal{M},X) \qquad \text{if } X \in \mathcal{M} \text{ and } V = \mathrm{Tan}(\mathcal{M},X).$$

Here of course $G(m,N)$ is the Grassmannian of m-planes in $\mathbf{R}^N$. It may seem odd to regard $\beta(X,V)$ as a function of X and V when the defining expression involves only X. However, it is natural and convenient to do so, as will be explained presently.

Note that the equation for problem (1) is

$$\beta(\mathcal{M})(X,V) = \nu(V), \tag{*}$$

where $\nu(V)$ is the unit normal to the oriented plane V. In other words, a flow $\mathcal{M}$ is a solution to problem 1 if and only if (*) holds for all (X,V) in the domain of $\beta(\mathcal{M})$.

Similarly, the equation for problem (2) is

$$\beta(\mathcal{M})(X,V) = mX,$$

and the equation for problem (3) is

$$\beta(\mathcal{M})(X,V) = -\operatorname{trace} \mathrm{II}(x)|V,$$

where $X = (x,t)$ and $\mathrm{II}(x)$ is the second fundamental form of S at x. Of course in problem (2), $\mathcal{M}$ should be contained in $\mathbf{S}^{N-1} \times \mathbf{R}$, and in problem (3) it should be contained in $S \times \mathbf{R}$.

In each of these three examples, note that $\beta(\mathcal{M})(X,V)$ is a Lipschitz (indeed smooth) function of X and V, with a Lipschitz constant that does not depend on $\mathcal{M}$. If, however, the quantity

$$\mathrm{v}(\mathcal{M},X) - H(\mathcal{M},X)$$

were regarded as a function of X alone (and not V), then, except in the second example, the Lipschitz constant would depend on the particular flow $\mathcal{M}$. For that reason, we choose to regard the Brakke operator as a function of position and tangent plane direction.

Since $\beta(\mathcal{M})$ is a function from a metric space to $\mathbf{R}^N$, we can define Hölder norms in the usual way. In particular, we let:

$$\begin{aligned}\|\beta(\mathcal{M})\|_0 &= \sup |\beta(\mathcal{M})(\cdot,\cdot)| \\ &= \sup_{X\in\mathcal{M}} |\beta(\mathcal{M})(X, \operatorname{Tan}(\mathcal{M},X))|\end{aligned}$$

and

$$(4) \qquad [\beta(\mathcal{M})]_\alpha = \sup \frac{|\beta(\mathcal{M})(X,\operatorname{Tan}(\mathcal{M},X)) - \beta(\mathcal{M})(Y,\operatorname{Tan}(\mathcal{M},Y))|}{\|X-Y\|^\alpha + \|\operatorname{Tan}(\mathcal{M},X) - \operatorname{Tan}(\mathcal{M},Y)\|^\alpha},$$

where the sup is over all $X \neq Y$ in $\mathcal{M}$. Finally, we let

$$\|\beta(\mathcal{M})\|_{0,\alpha} = \|\beta(\mathcal{M})\|_0 + [\beta(\mathcal{M})]_\alpha.$$

It is also useful to have scale invariant versions. If $\mathcal{M}$ is a proper flow in U, we let

$$d(\mathcal{M};U) = \sup_{X\in\mathcal{M}} d(X,U) = \sup_{X\in\mathcal{M}} \inf_{Y\notin U} \|X-Y\|.$$

We will assume that $d(\mathcal{M};U) < \infty$. Dilate $\mathcal{M}$ and U by $1/d(\mathcal{M};U)$ to get $\mathcal{M}'$ and U' with $d(\mathcal{M}';U') = 1$. We define

$$\begin{aligned}\|\beta(\mathcal{M})\|_{0;U} &= \|\beta(\mathcal{M}')\|_0, \\ [\beta(\mathcal{M})]_{\alpha;U} &= [\beta(\mathcal{M}')]_\alpha, \\ \|\beta(\mathcal{M})\|_{0,\alpha;U} &= \|\beta(\mathcal{M}')\|_{0,\alpha}.\end{aligned}$$

Of course one can also define these scale invariant quantities directly by modifying the definitions of the non-invariant versions. For instance

$$\|\beta(\mathcal{M})\|_{0;U} = d(\mathcal{M};U) \sup \|\beta(\mathcal{M})(\cdot,\cdot)\|.$$

Similarly, to define $[\beta(\mathcal{M})]_{\alpha;U}$, one modifies (4) by multiplying the numerator by $d = d(\mathcal{M};U)$ and dividing the term $\|X-Y\|^\alpha$ by d^α. It is then straightforward to check that

$$(5) \qquad \|\beta(\mathcal{M})\|_{0,\alpha} \le \frac{\|\beta(\mathcal{M})\|_{0,\alpha;U}}{d(\mathcal{M};U)} \qquad \text{if } d(\mathcal{M};U) \ge 1.$$

4.1. Theorem. *Let $\epsilon \in (0,\bar{\epsilon})$, where $\bar{\epsilon}$ is as in Theorem 3.2, and let $\alpha \in (0,1)$. There is a $C = C(N,m,\alpha,\epsilon) < \infty$ with the following property. Suppose $\mathcal{M}$ is a proper $C^{2,\alpha}$ flow in $U \subset \mathbf{R}^{N,1}$ such that*

$$\Theta(\mathcal{M},X,r) \le 1+\epsilon$$

for all $X \in \mathcal{M}$ and $0 < r < d(U,X)$. Then

$$K_{2,\alpha;U}(\mathcal{M}) < C(1 + \|\beta(\mathcal{M})\|_{0,\alpha;U}).$$

Proof. The proof is almost the same as the proof of Theorem 3.1. We assume the theorem is false, and we get a sequence of flows $\mathcal{M}_i$ in U_i and points $X_i \in \mathcal{M}_i$ such that

$$\Theta(\mathcal{M}_i, X, r) \le 1 + \epsilon$$

for all $X \in \mathcal{M}_i$ and $0 < r < d(X, U_i)$, and such that

$$\frac{K_{2,\alpha;\,U_i}(\mathcal{M}_i)}{1 + \|\beta(\mathcal{M}_i)\|_{0,\alpha;U_i}} \to \infty. \tag{6}$$

As in section 3.1, may assume that

$$s_i = K_{2,\alpha;\,U_i}(\mathcal{M}_i) < \infty$$

for each i. By translating and dilating suitably, we may also assume that

$$d(0, U_i)\, K_{2,\alpha}(\mathcal{M}_i, 0) \ge \frac{1}{2} s_i \tag{7}$$

and that

$$K_{2,\alpha}(\mathcal{M}_i, 0) = 1. \tag{8}$$

As before, this implies that

$$d(\mathcal{M}_i, U_i) \ge d(0, U_i) \to \infty \tag{9}$$

and also (after passing to a subsequence) that the $\mathcal{M}_i$ converge in C^2 to a flow $\mathcal{M}$ that is proper in all of $\mathbf{R}^{N,1}$.

Now by (6), (7), and (8),

$$\frac{\|\beta(\mathcal{M}_i)\|_{0,\alpha;U_i}}{d(\mathcal{M}_i; U_i)} \to 0,$$

which implies by (5) and (9) that

$$\|\beta(\mathcal{M}_i)\|_{0,\alpha} \to 0. \tag{10}$$

In particular, $\sup |\,\mathrm{v}(\mathcal{M}_i, \cdot) - \mathrm{H}(\mathcal{M}_i, \cdot)| \to 0$ Thus the limit flow $\mathcal{M}$ is a mean curvature flow.

The rest of the proof is exactly as in section 3.1, except that the u_i are no longer solutions of the mean curvature flow equation, but rather satisfy

$$\frac{\partial}{\partial t} u_i - \Delta u_i = f_i + \pi' \beta_i - \mathrm{D}u_i(x, t) \circ \pi \beta_i, \tag{11}$$

where the right hand side is as follows. First, f_i is exactly as in the proof of 3.1. Second,

$$\pi : \mathbf{R}^N \cong \mathbf{R}^m \times \mathbf{R}^{N-m} \to \mathbf{R}^m$$
$$\pi' : \mathbf{R}^N \cong \mathbf{R}^m \times \mathbf{R}^{N-m} \to \mathbf{R}^{N-m}$$

are the orthogonal projections. Third,

$$\beta_i(x, t) = \beta(\mathcal{M}_i)((x, u(x, t), t), \mathrm{Tan}(\mathcal{M}_i, (x, u(x, t), t))). \tag{12}$$

(See section 8.4 in the appendix for derivation of these formulas.)

Note that the Tan part of (12) depends only on $\mathrm{D}u_i(x, t)$, and that the dependence is smooth. Thus β_i is essentially (modulo slight abuse of notation) the composition of

(1) the map $\beta(\mathcal{M}_i)$, and
(2) the function $(x, t) \mapsto (x, u_i(x), t, \mathrm{D}u_i(x, t))$.

The first converges to 0 in C^α (by (10)), and the second converges to 0 in C^1 on compact sets. Thus the composition β_i converges to 0 in C^α on compact sets.

Hence $\pi'\beta_i$, $\pi\beta_i$, and therefore $\mathrm{D}u_i \circ \pi\beta_i$ also converge to 0 in $C^{0,\alpha}$ on compact sets.

The f_i converge to 0 in C^α on compact sets as in section 3.1.

Thus the Schauder estimates (§8.2) for the heat equation (11) imply that the u_i converge to 0 in $C^{2,\alpha}$ on compact sets, which contradicts the normalization $K_{2,\alpha;\,U_i}(\mathcal{M}_i) \equiv 1$. $\square$

4.2. Corollary. *Suppose that M is a compact manifold and that*

$$F : M \times [0, T) \to \mathbf{R}^{N,1}$$

is a classical solution to one of the problems (1)-(3) *mentioned at the beginning of this section. Let $\mathcal{M}$ be the spacetime flow swept out by F. Suppose $X = (x, T)$ is a point in $\overline{\mathcal{M}}$ such that*

$$\Theta(\mathcal{M}, X) < 1 + \overline{\epsilon}.$$

Then there is a spacetime neighborhood U of X such that

$$\overline{\mathcal{M}} \cap U$$

is a $C^{2,\alpha}$ flow.

Proof. Choose $\epsilon < \overline{\epsilon}$ with

$$\Theta(\mathcal{M}, X) < 1 + \epsilon.$$

By continuity and monotonicity (§2.9), there is an $r > 0$ and a spacetime neighborhood U of X such that

$$\Theta(\mathcal{M}, Y, \rho) < 1 + \epsilon$$

for all $Y \in U$ and $0 < \rho \le r$. We may choose U to have diameter $< r$. Let $T_i < T$ converge to T, and let

$$\mathcal{M}_i = \mathcal{M} \cap U \cap \{Y : \tau(Y) \le T_i\}.$$

Then by Theorem 4.1,

$$K_{2,\alpha;U}(\mathcal{M}_i) \le C(1 + \|\beta(\mathcal{M}_i)\|_{0,\alpha;\,U}).$$

Letting $i \to \infty$ gives the same bound for $K_{2,\alpha;U}(\overline{\mathcal{M}} \cap U)$. $\square$

5. Edge Behavior

Suppose that M is a compact manifold with boundary and that

$$F : M \times (a, b) \to \mathbf{R}^N$$

is a smooth map such that each $F(\cdot, t)$ is an embedding. Let $\mathcal{M}$ be the set in spacetime traced out by F:

$$\mathcal{M} = \{(F(x, t), t) : t \in (a, b)\}.$$

Then $\mathcal{M}$ is not a smooth flow because it is not a manifold in $\mathbf{R}^{N+1}$, but rather a manifold with boundary, the boundary being

$$\mathcal{N} = \{(F(x,t) : x \in \partial \mathcal{M}, t \in (a,b)\}.$$

Thus it is useful to enlarge the concept of flow to "flow with edge". (Calling $\mathcal{N}$ the edge rather than the boundary distinguishes it from other kinds of boundary.)

In general, suppose $\mathcal{M}$ is a C^1 submanifold-with-boundary in $\mathbf{R}^{N+1}$, and suppose that the time function τ has no critical points on $\mathcal{M}$ or on the boundary of $\mathcal{M}$. (In other words, the restrictions of τ to $\mathcal{M}$ and to its boundary have no critical points.) Then we will call $\mathcal{M}$ a **fully regular flow-with-edge**. The boundary of $\mathcal{M}$ will then be called the edge of $\mathcal{M}$ and will be denoted $\mathcal{E}\mathcal{M}$.

To allow for sudden vanishing, if $\mathcal{M}$ is a fully regular flow-with-edge and $T \in (-\infty, \infty]$, then the truncated set

$$\mathcal{M}' = \{X \in \mathcal{M} : \tau(X) \leq T\}$$

will be called a **regular flow-with-edge**, the edge being

$$\mathcal{E}\mathcal{M}' = \{X \in \mathcal{E}\mathcal{M} : \tau(X) \leq T\}.$$

Note that the edge $\mathcal{E}\mathcal{M}'$ is itself a regular flow (without edge) of one lower dimension.

To illustrate these concepts, suppose M is the half open interval from a to b in $\mathbf{R}^N$, with $a \in M$ and $b \notin M$. Let $\mathcal{M} = M \times (0,1]$. Then $\mathcal{M}$ is a smooth flow with edge. The edge $\mathcal{E}\mathcal{M}$ is $[a] \times (0,1]$. The set $\overline{\mathcal{M}} \setminus \mathcal{M}$, which is a kind of boundary of $\mathcal{M}$, is something quite different, namely the union of $M \times [0]$ and $[b] \times (0,1]$.

For flows with edge, it is useful to introduce modified Gaussian density ratios. If $\mathcal{M}$ is a regular flow-with-edge and X is a point in spacetime, let

$$\mathcal{C}_X\mathcal{M} = \{X + \mathcal{D}_\lambda(Y - X) \colon Y \in \mathcal{E}\mathcal{M},\ \lambda > 0\},$$

and let

$$\Theta^*(\mathcal{M}, X, r) = \begin{cases} \Theta(\mathcal{M}, X, r) + \Theta(\mathcal{C}_X\mathcal{M}, X, r) & \text{if } X \notin \mathcal{E}\mathcal{M}, \\ \Theta(\mathcal{M}, X, r) + \Theta(\mathcal{C}_X\mathcal{M}, X, r) + \frac{1}{2} & \text{if } X \in \mathcal{E}\mathcal{M}. \end{cases}$$

We also let

$$(*) \qquad \Theta^*(\mathcal{M}, X) = \lim_{r \to 0} \Theta^*(\mathcal{M}, X, r).$$

$$\Theta^*(\mathcal{M}, \infty) = \lim_{r \to \infty} \Theta^*(\mathcal{M}, X, r).$$

provided the limits exist (and, in the case of $\Theta^*(\mathcal{M}, \infty)$, provided the limit is independent of X).

Then $\Theta^*(\mathcal{M}, X, r)$ has most of the same properties for flow-with-edge that the ordinary Gaussian density ratio $\Theta(\mathcal{M}, X, r)$ has for flows. For example, $\Theta^*(\mathcal{M}, X, r)$ is continuous in X and r, and if $\mathcal{M}$ is a smooth mean curvature flow-with-edge that is proper in in $\mathbf{R}^N \times (a,b)$, then

$$\Theta^*(\mathcal{M}, X, r)$$

is an increasing function of r for $0 < r < \sqrt{\tau(X) - a}$.

Slightly weaker statements are true for flows-with-edge that are proper in other open sets U and for which $|\,\mathrm{v}(\mathcal{M}, \cdot) - \mathrm{H}(\mathcal{M}, \cdot)|$ is bounded. For such flows, the limit (*) does exist for all X, and

$$\Theta^*(\mathcal{M}, X) = \begin{cases} \theta(\mathcal{M}, X) & \text{if } X \notin \mathcal{E}\mathcal{M} \\ \theta(\mathcal{M}, X) + \frac{1}{2} & \text{if } X \in \mathcal{E}\mathcal{M} \end{cases}.$$

5.1. Lemma. *Suppose $\mathcal{M}$ is a smooth proper mean curvature flow-with-edge in $\mathbf{R}^{N,1}$. Suppose the edge has the form*

$$\mathcal{E}\mathcal{M} = V \times (-\infty, T]$$

for some $T \le \infty$, where V is a linear subspace of $\mathbf{R}^N$. Suppose also that

$$\Theta^*(\mathcal{M}, \infty) < 1 + (\overline{\epsilon}/2),$$

where $\overline{\epsilon}$ is as in Theorem 3.2. Then after a suitable rotation, $\mathcal{M}$ has the form

$$H \times [0]^{N-m} \times (-\infty, T],$$

where $H = \{x \in \mathbf{R}^m : x_m \ge 0\}$.

Proof. Let

$$\mathcal{M}' = \mathcal{M} \cup \{(-x, t) : (x, t) \in \mathcal{M}\}.$$

Then $\mathcal{M}'$ is a smooth proper mean curvature flow (with no edge), and

$$\begin{aligned}\Theta(\mathcal{M}', \infty) &= 2\Theta(\mathcal{M}, \infty) \\ &= 2(\Theta^*(\mathcal{M}, \infty) - \frac{1}{2}) \\ &< 1 + \overline{\epsilon}.\end{aligned}$$

The result follows immediately from Theorem 3.2. □

5.2. Theorem. *Let $\epsilon \in (0, \overline{\epsilon}/2)$, where $\overline{\epsilon}$ is as in Theorem 3.2. For every $0 < \alpha < 1$, m, and N, there exists a number $C = C(N, m, \alpha, \epsilon) < \infty$ with the following property. Suppose $\mathcal{M}$ is a $C^{2,\alpha}$ flow-with-edge in $\mathbf{R}^{N,1}$ such that*

$$\Theta^*(\mathcal{M}, X, r) \le 1 + \epsilon$$

for all $X \in \mathcal{M}$ and $0 < r < d(U, X)$. Then

$$K_{2,\alpha;\,U}(\mathcal{M}) \le C\left(1 + \|\beta(\mathcal{M})\|_{0,\alpha;\,U} + K_{2,\alpha;\,U}(\mathcal{E}\mathcal{M})\right).$$

Proof. As in sections 3.1 and 4.1, we assume that the theorem is false, and we get a sequence of proper flows $\mathcal{M}_i$ in open sets U_i such that

$$\sup_{X \in \mathcal{M}_i,\, 0 < r < d(X, U_i)} \Theta^*(\mathcal{M}_i, X, r) \le 1 + \epsilon$$

and such that

$$\frac{K_{2,\alpha;\,U_i}(\mathcal{M}_i)}{1 + \|\beta(\mathcal{M}_i)\|_{0,\alpha;U_i} + K_{2,\alpha;\,U_i}(\mathcal{E}\mathcal{M}_i)} \to \infty.$$

As before, by suitably translating and rotating, we may assume that $0 \in \mathcal{M}_i$, and that

$$\begin{aligned}&K_{\mathcal{M}_i,0} = 1 \\ &d(0, U_i) = d(0, U_i)\, \mathrm{K}_{2,\alpha}(\mathcal{M}_i, 0) > \frac{1}{2} K_{2,\alpha;\,U_i}(\mathcal{M}_i).\end{aligned}$$

Thus

$$(1) \qquad \frac{1 + \|\beta(\mathcal{M}_i)\|_{0,\alpha;U_i} + K_{2,\alpha;\,U_i}(\mathcal{E}\mathcal{M}_i)}{d(0,U_i)} \to 0.$$

From (1),

$$\frac{K_{2,\alpha;\,U_i}(\mathcal{E}\mathcal{M}_i)}{d(0,U_i)} \to 0.$$

This implies that

$$(2) \qquad K_{2,\alpha}(\mathcal{E}\mathcal{M}_i, \cdot) \to 0$$

uniformly on compact sets in spacetime (because $d(0,U_i) \to \infty$). And as before, the functions $K_{2,\alpha}(\mathcal{M}_i, \cdot)$ are uniformly bounded on compact subsets of $\mathbf{R}^{N,1}$. Thus by passing to a subsequence, we may assume that the $\mathcal{M}_i$ converge in C^2 on compact sets to a proper flow-with-edge $\mathcal{M}$ in $\mathbf{R}^{N,1}$. As in section 4.1, $\beta(\mathcal{M}) \equiv 0$.

By (2) and the Arzela-Ascoli Theorem 2.7, $K_{2,\alpha}(\mathcal{E}\mathcal{M}, \cdot) \equiv 0$, so the edge $\mathcal{E}\mathcal{M}$ must be one of the following:

(1) the emptyset, or
(2) $V \times (-\infty, T]$ where V is an $(m-1)$ dimensional subspace of $\mathbf{R}^N$ and $T \in [0, \infty]$.

In case (1), the rest of the proof is just as in section 4.1. In case (2), the hypotheses of Lemma 5.1 are satisfied, so $\mathcal{M}$ has the form

$$H \times [0]^{N-m} \times (-\infty, T]$$

asserted by the lemma.

The rest of the proof is exactly as in section 4.1, except that now we now apply Schauder estimates at the boundary (§8.3) on a sequence of domains in $\mathbf{R}^{m,1}$ converging to $H \times (-\infty, T]$. □

5.3. Corollary. *Let M be a compact m-manifold with boundary and let*

$$\phi : \partial M \times (0, T] \to \mathbf{R}^N$$

be a smooth 1-parameter family of embeddings. Suppose

$$F : M \times (0, T) \to \mathbf{R}^N$$

is a smooth 1-parameter family of embeddings such that

$$F(x,t) = \phi(x,t) \text{ for all } x \in \partial M \text{ and } t \in (0,T)$$

and such that

$$\left(\frac{\partial F(x,t)}{\partial t} \right)^{\perp}$$

is equal to the mean curvature of $F(M,t)$ at $F(x,t)$ for all $x \in M$ and $0 < t < T$. Let $\mathcal{M}$ be the set swept out by F:

$$\mathcal{M} = \{(F(x,t),t) : x \in \mathcal{M}, 0 < t < T\}.$$

If $X = (x, T)$ is a point in $F(\partial M, T)$ such that

$$\Theta^*(\mathcal{M}, X) < 1 + \bar{\epsilon}/2,$$

then there is a spacetime neighborhood U of X such that

$$\overline{\mathcal{M}} \cap U$$

is a smooth proper flow in U.

The proof is exactly like the proof of Theorem 3.5. Of course the result is also true, with the same proof, for mean curvature flows in Riemannian manifolds, or, more generally, for flows whose Brakke operators are Hölder continuous functions of position, time, and tangent plane direction.

6. Bounded Additional Forces

Consider a surface moving by mean curvature plus a bounded measurable function. For example, this is the case for motion by mean curvature with smooth (or even $C^{1,1}$) obstacles. In this section, we will consider the Brakke operator to be a function of position only:

$$\begin{aligned} &\beta(\mathcal{M}) : \mathcal{M} \to \mathbf{R}^N \\ &\beta(\mathcal{M})(X) = \mathrm{v}(\mathcal{M}, X) - \mathrm{H}(\mathcal{M}, X). \end{aligned}$$

Let $p > m$. We define $\kappa_{2,p}(\mathcal{M}, X)$ just as we defined $K_{2,\alpha}(\mathcal{M}, X)$ in section 2.5, except that in the definition, we replace the parabolic $C^{2,\alpha}$ norm of u by the parabolic $W^{2,p}$ norm:

$$\|u\|_{W^{2,p}} = \left(\int (|u|^p + |\partial_t u|^p + |\,\mathrm{D}^2 u|^p) \right)^{1/p}.$$

From $\kappa_{2,p}(\mathcal{M}, \cdot)$ we also define $\kappa_{2,p;\,U}(\mathcal{M})$ just as $K_{2,\alpha\,;U}(\mathcal{M})$ was defined from $K_{2,\alpha}(\mathcal{M}, \cdot)$ in section 2.5.

We define $W^{2,p}$ flows just as we defined regular flows in section 2.2, except that instead of requiring $\mathcal{M}$ to be C^1 as a submanifold of $\mathbf{R}^{N+1}$, we require that $\kappa_{2,p}(\mathcal{M}, \cdot)$ be everywhere finite.

6.1. Theorem. *Let $\epsilon \in (0, \bar{\epsilon}/2)$, where $\bar{\epsilon}$ is as in section 3.2. For every $p > m$, there is a number $C = C(N, m, \epsilon, p) < \infty$ with the following property. Suppose $\mathcal{M}$ is a proper $W^{2,p}$ flow-with-edge in $U \subset \mathbf{R}^{N,1}$ such that*

$$\sup_{X \in \mathcal{M},\, 0 < r < d(X,U)} \Theta^*(\mathcal{M}, X, r) \le 1 + \epsilon.$$

Then

$$\kappa_{2,p\,;\,U}(\mathcal{M}) + K_{1,\alpha\,;U}(\mathcal{M}) \le C\,(1 + \|\beta(\mathcal{M})\|_{0;U} + K_{1,1\,;U}(\mathcal{E}\mathcal{M}))\,,$$

where $\alpha = 1 - \frac{m}{p}$.

Remark. Since $\beta(\mathcal{M})(\cdot)$ is only defined almost everywhere, the term $\|\beta(\mathcal{M})\|_{0;U}$ means $d(\mathcal{M}, U)$ times the essential supremum of $|\beta(\mathcal{M})(\cdot)|$.

Proof. We describe the proof for flows without edge, since the edge is handled just as it was in section 5.

A theorem of Morrey [L §6.8] bounds the $C^{1,\alpha}$ norm of a function on one domain in terms of the $W^{2,p}$ norm on any larger domain. Thus since $\mathcal{M}$ is a $W^{2,p}$ flow, it is also a $C^{1,\alpha}$ flow.

Define a new quantity $J_{2,p}(\mathcal{M}, X)$ exactly like $K_{2,\alpha}(\mathcal{M}, X)$ except that, instead of the $C^{2,\alpha}$ norm of u in section 2.5, we use the norm:

$$\|u\|_{W^{2,p}} + \|u\|_{C^{1,\alpha}}$$

We now copy the proof of Theorem 4.1, using $J_{2,p}$ instead of $K_{2,\alpha}$. As in that proof, we get a sequence $\mathcal{M}_i$ in U_i such that

(1) $\frac{d(0,U_i)}{1+\|\beta\|_{0;U_i}} \to 0$,
(2) $\mathcal{M}_i$ converges (after a rotation) locally in C^1 to $\mathbf{R}^m \times [0]^{N-m} \times (-\infty, T]$,
(3) $J_{2,p}(\mathcal{M}_i, 0) \equiv 1$, and
(4) $J_{2,p}(\mathcal{M}_i, \cdot)$ is uniformly bounded as $i \to \infty$ on compact subsets of spacetime.

As in section 4.1, we then get functions u_i such that

(5) u_i converges to 0 in C^1 by (2) and (4), and
(6) the u_i are uniformly bounded as $i \to \infty$ in $W^{2,p}$ on compact subsets of spacetime (by (4)).

As before (see (11) in section 4.1), the u_i satisfy the equation

$$(*) \qquad \frac{\partial}{\partial t} u_i - \Delta u_i = f_i + \pi' \beta_i - Du_i \circ \pi\, \beta_i,$$

where

$$f_i = -\sum_{j,k} \frac{\mathrm{D}_j u_i \, \mathrm{D}_k u_i}{1 + |\,\mathrm{D} u_i|^2} \, \mathrm{D}_{jk} u_i.$$

Now by (5) and (6), the f_i tend to 0 in $W^{2,p}$ on compact subsets of spacetime. Also,

$$\|\beta_i(\cdot)\|_0 \leq \frac{\|\beta(\mathcal{M}_i)\|_{0;U_i}}{d(0, U_i)},$$

which tends to 0 by (2). That is, β_i tends to 0 uniformly. Hence the entire right side of (*) tends to 0 in L^p. By the L^p estimates [L, VII §4-5] for the heat equation, the u_i tend to 0 in $W^{2,p}$ on compact subsets of spacetime. By Morrey's Theorem [L §7.8], they also tend to 0 in $C^{1,\alpha}$ on compact subsets of spacetime. But this contradicts (3), proving the theorem. □

7. Mean Curvature Flow of Varifolds

In this section, we briefly indicate how the regularity theory developed in this paper can also apply outside the context of classical mean curvature flows, in particular to certain mean curvature flows of varifolds. Such varifold flows were introduced by Brakke [B] and are now often called "Brakke flows". Sections 6 and 7 of Ilmanen's booklet [I1] give a nice introduction to Brakke flows.

If μ is a Radon measure on $\mathbf{R}^N$, let

$$\Gamma_m(\mu, x, r) = \frac{1}{(4\pi r^2)^{m/2}} \int_{\mathbf{R}^N} \exp\left(\frac{-|y-x|^2}{4r^2}\right) d\mu(y).$$

Note this is the Gaussian density that occurs in the monotonicity formula: if $\mathcal{M}$ is a m-dimensional mean curvature flow in $\mathbf{R}^N$, then

$$\Theta(\mathcal{M}, X, r) = \Gamma_m(\mathcal{M}(\tau(X) - r^2), X, r)$$

where $\tau(X)$ is the time coordinate of the spacetime point X and $\mathcal{M}(t)$ is the Radon measure in $\mathbf{R}^N$ determined by the flow at time t.

Fix a number $\lambda < \infty$. Let $\mathcal{V}(\lambda, m, N)$ be the class of all Radon measures in $\mathbf{R}^N$ such that all m-dimensional Gaussian density ratios are $\leq \lambda$:

$$\sup_{x,r} \Gamma_m(\mu, x, r) \leq \lambda \qquad (x \in \mathbf{R}^N, \lambda > 0).$$

Of course by the monotonicity formula, this class is preserved by mean curvature flow.

7.1. Lemma. *If $\mu_i \in \mathcal{V}(\lambda, m, N)$ converges weakly to μ, then $\Gamma_m(\mu_i, \cdot, \cdot)$ converges uniformly on compact subsets of $\mathbf{R}^N \times (0, \infty)$ to $\Gamma_m(\mu, \cdot, \cdot)$. Thus $\mu \in \mathcal{V}(\lambda, m, N)$.*

Proof. This may be proved directly. Or one may observe that the function

$$u(x,t) = \left(4\pi\sqrt{t}\right)^{(m-N)/2} \Gamma_m\left(x, \sqrt{t}\right)$$

is the solution to the heat equation with initial values given (distributionally) by the measure μ. The conclusions then follow from standard facts about the heat equation. $\square$

Now let $\mathcal{S}(\lambda, m, N)$ denote the class of Brakke flows $\mathcal{M}$ of m-dimensional surfaces in $\mathbf{R}^N$ with the following properties:

(1) $\mathcal{M}$ is defined on the time interval $[0, \infty)$.
(2) $\mathcal{M}(t) \in \mathcal{V}(\lambda, m, N)$ for all t.
(3) If $\Theta(\mathcal{M}, X) < 1 + \bar{\epsilon}$ (where $\bar{\epsilon}$ is as in theorem 3.2) and if $\tau(X) > 0$, then X is a fully smooth point of $\mathcal{M}$. That is, there is a spacetime neighborhood U of X such that $U \cap \operatorname{spt}(\mathcal{M})$ is a fully smooth flow (in the sense of §2.2).

The support $\operatorname{spt}(\mathcal{M})$ of $\mathcal{M}$ is the set of smallest closed set in spacetime that contains (x,t) for every x in the support of $\mathcal{M}(t)$. For $X = (x,t)$ with $t > 0$, $X \in \operatorname{spt} \mathcal{M}$ if and only if $\Theta(\mathcal{M}, X) > 0$ or, equivalently, $\Theta(\mathcal{M}, X) \geq 1$.

7.2. Theorem. *The class $\mathcal{S}(\lambda, m, N)$ is compact.*

Proof. The class of flows satisfying the first two conditions is compact [I1 7.1], so we need only verify closure of the subclass satisfying the third condition. Closure is proved in the following theorem. $\square$

7.3. Theorem. *Suppose $\mathcal{M}_i \in \mathcal{S}(\lambda, m, N)$ converge as Brakke flows to $\mathcal{M}$. Suppose X is a spacetime point with $\tau(X) > 0$ such that $\Theta(\mathcal{M}, X) < 1 + \bar{\epsilon}$. Then there is a spacetime neighborhood U of X and an integer $I < \infty$ such that*

(1) *$U \cap \operatorname{spt}(\mathcal{M}_i)$ and $U \cap \operatorname{spt} \mathcal{M}$ are fully smooth for $i \geq I$, and*
(2) *$U \cap \operatorname{spt}(\mathcal{M}_i)$ converges smoothly to $U \cap \operatorname{spt}(\mathcal{M})$ on compact subsets of U.*

Proof. By definition, $\Theta(\mathcal{M}, X, R) < 1 + \epsilon$ for some R with $0 < R < \sqrt{\tau(X)}$ and for some $\epsilon < \bar{\epsilon}$. By the lemma,

$$(*) \qquad \sup_{Y \in U} \Theta\left(\mathcal{M}_i, Y, \sqrt{R^2 + \tau(Y) - \tau(X)}\right) < 1 + \epsilon$$

will hold for some spacetime neighborhood U of X for all sufficiently large i.

The conclusions then follow immediately from theorems 2.6 and 3.1. (See also the remark after theorem 3.1.) □

Remark. These theorems remain true (with essentially the same proofs) if in the definition of $\mathcal{S}(\lambda, m, N)$, "fully smooth" is replaced by "smooth".

7.4. Theorem. *Let M be a compact smooth embedded hypersurface in $\mathbf{R}^{m+1}$. Then there is a $\lambda < \infty$ and a Brakke flow $\mathcal{M} \in \mathcal{S}(\lambda, m, N)$ such that $\mathcal{M}(0)$ is the Radon measure associated with M.*

Sketch of Proof. Let $u : \mathbf{R}^{m+1} \to \mathbf{R}$ be a smooth function such that:

(1) $u^{-1}(0) = M$,
(2) ∇u does not vanish anywhere on M,
(3) $u(x) = |x|$ for all sufficiently large $|x|$.

(Condition (3) is somewhat arbitrary: it could be replaced by any other condition asserting that u is reasonably well behaved at infinity.)

The graph G_k of $ku(\cdot)$ is a hypersurface in $\mathbf{R}^{m+2}$, and there is a classical mean curvature flow, with initial surface G_k, that is smooth for all time $t \geq 0$. Let $\mathcal{M}_k$ be the associated Brakke flow. Note there is a $\lambda < \infty$ such $\mathcal{M}_k \in \mathcal{V}(\lambda, m+1, m+2)$ for all k. Thus a subsequence of the $\mathcal{M}_k$ will converge to a limit $\mathcal{M}'$ in $\mathcal{V}(\lambda, m+1, m+2)$.

We would like to say that $\mathcal{M}'$ is invariant under translations in the vertical (i.e., $\mathbf{e}_{n+1}$) direction. This is not always true. However, we can always get a translationally invariant flow as follows. Translate $\mathcal{M}'$ by $-j$ in the $\mathbf{e}_{m+2}$ direction to get a new flow $\mathcal{M}'_j$. Now let $\mathcal{M}^*$ be a subsequential limit of the $\mathcal{M}'_j$'s as $j \to \infty$. Then one can show that $\mathcal{M}^*$ is translationally invariant.

The translational invariance means that $\mathcal{M}^*$ is the Cartesian product with $\mathbf{R}$ of a flow $\mathcal{M}$ of m-dimensional surfaces in $\mathbf{R}^{m+1}$. Since $\mathcal{M}^*$ is in $\mathcal{S}(\lambda, m+1, m+2)$, it follows easily that $\mathcal{M}$ is in $\mathcal{S}(\lambda, m, m+1)$. □

Remark. We could have used a subsequential limit of the $\mathcal{M}'_j$ as $j \to -\infty$. The resulting flow $\mathcal{M}_*$ is also translationally invariant, though it may in general differ from $\mathcal{M}^*$. (Indeed, $\mathcal{M}_*$ and $\mathcal{M}^*$ differ if and only if the initial surface M "fattens" under mean curvature flow.)

8. Appendix

In this section, we prove the Arzela-Ascoli Theorem §2.7, we state the Schauder estimates for the heat equation, and we derive the non-parametric equations for mean curvature flow. First we recall the definitions of the parabolic Hölder norms. Suppose W is an open subset of the spacetime $\mathbf{R}^{n,1}$. If u is a map from W to a Euclidean space, then

$$[u]_\alpha = \sup_{X,Y\in W,\, X\neq Y} \frac{|u(X)-u(Y)|}{\|X-Y\|^\alpha},$$
$$\|u\|_{0,\alpha} = \sup_{X\in W} |u(X)| + [u]_\alpha.$$

Of course $\|X-Y\|$ denotes the parabolic distance from X to Y.

If p is an nonnegative integer and $0<\alpha<1$, we let

$$\|u\|_{p,\alpha} = \sum_{j+2k\leq p} \left\| \mathrm{D}^j(\partial_t)^k u\right\|_{0,\alpha}.$$

Here D denotes the derivative with respect to the spatial variables:

$$D = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right),$$

where $(x,t) = (x_1,\dots,x_m,t)$ are the coordinates in spacetime.

8.1. Theorem. *Suppose $\mathcal{M}_i$ are sets in $\mathbf{R}^{N,1}$ that converge to $\mathcal{M}$ as sets. Suppose that $X_i\in\mathcal{M}_i$ converges to $X\in\mathcal{M}$. Then*

$$K_{2,\alpha}(\mathcal{M},X) \leq \liminf K_{2,\alpha}(\mathcal{M}_i,X_i).$$

Proof. By passing to a subsequence, we may assume that the liminf is a limit. We may also assume that the limit is a finite number L, as otherwise the result is vacuously true. By scaling, we may assume that $L=1$.

We may also assume that $X=0$. Translate $\mathcal{M}_i$ by $-X_i$ and then dilate by $K_{2,\alpha}(\mathcal{M}_i,X_i)$ to get a new set $\mathcal{M}_i'$. Note that the $\mathcal{M}_i'$ also converge as sets to $\mathcal{M}$, and that

$$K_{2,\alpha}(\mathcal{M}_i',0) = 1.$$

Thus there are rotations ϕ_i of $\mathbf{R}^N$ such that

$$\tilde{\phi}_i(\mathcal{M}_i') \subset \operatorname{graph}(u_i),$$

where

$$u_i : \mathbf{B}^{m,1}\to\mathbf{R}^{N-m}, \qquad \|u_i\|_{2,\alpha}\leq 1,$$

and $\tilde{\phi}(x,t) := (\phi x, t)$ is the isometry of spacetime induced by ϕ. (Here $\mathbf{B}^{m,1} = \mathbf{B}^m\times(-1,1)$ is the unit ball in the spacetime $\mathbf{R}^{m,1}$.) By passing to a subsequence, we may assume that the ϕ_i converge to a rotation ϕ and that the u_i converge uniformly to a function

$$u:\mathbf{B}^{m,1}\to\mathbf{R}^{N-m}$$

with $\|u\|_{2,\alpha}\leq 1$.

It follows (since $\mathbf{B}$ is the *open* ball) that

$$\tilde{\phi}\mathcal{M}\cap\mathbf{B}\subset\operatorname{graph}(u)$$

and therefore that $K_{2,\alpha}(\mathcal{M},0)\leq 1 = L$. □

The Arzela-Ascoli Theorem §2.7 is an easy consequence of Theorem 8.1.

Remark. The same proof shows that Theorem 8.1 is true for any other norm $\|\cdot\|_*$, provided the set of maps $u:\mathbf{B}^{m,1}\to\mathbf{R}^{N-m}$ with $\|u\|_*\leq 1$ is a compact subset of $C^0(\mathbf{B}^{m,1},\mathbf{R}^{N-m})$.

Schauder Estimates

In this section, L will denote the ordinary heat operator in $\mathbf{R}^{m,1}$:

$$Lu = u_t - \Delta u$$

8.2. Theorem. *Let $0 < r < r' < \infty$ and $0 \leq T \leq \infty$. Let*

$$\Omega = \{X \in \mathbf{R}^{m,1} : \|X\| < r, \quad \tau(X) \leq T\},$$
$$\Omega' = \{X \in \mathbf{R}^{m,1} : \|X\| < r', \quad \tau(X) \leq T\}.$$

If u is a $C^{2,\alpha}$ function on Ω', then

$$\|u|\Omega\|_{2,\alpha} \leq c(\|u\|_0 + \|Lu\|_{0,\alpha}),$$

where C may depend on r, R, m, and α.

In section 5, we also used the corresponding estimate at the boundary. For that estimate, suppose ϕ and ψ are $C^{2,\alpha}$ real-valued functions defined on $\mathbf{R}^{m-1,1}$. Let

$$K = K_\phi = \{(x,t) \in \mathbf{R}^{m,1} : x_m \geq \phi(x_1, \ldots, x_{m-1}),\, t \leq T\},$$

and let

$$\Omega = \{X \in K : \|X\| < r\},$$
$$\Omega' = \{X \in K : \|X\| < r'\}.$$

8.3. Theorem. *Suppose $u : \Omega' \to \mathbf{R}$ is a $C^{2,\alpha}$ function such that*

$$u(x, \phi(x), t) = \psi(x,t)$$

for all $x \in \mathbf{R}^{m-1}$ with $(x, \phi(x), t) \in \Omega$. Then

$$\|u|\Omega\|_{2,\alpha} \leq c(\|u\|_0 + \|Lu\|_{2,\alpha}, + \|\psi\|_{2,\alpha})$$

where c may depend on r, R, m, α, and $\|\phi\|_{2,\alpha}$.

Proofs of the Schauder estimates may be found in chapter IV of [L] and in chapters 3 and 4 of [F]. A short, elementary proof of a much more general form of the Schauder Estimates is given in [S].

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8.4. Theorem. *Let*

$$\mathcal{M} = \{(x, u(x,t), t) : (x,t) \in \Omega\}$$

be the graph of a smooth function $u : \Omega \to \mathbf{R}^{N-m}$ defined on an open subset Ω of $\mathbf{R}^{m,1}$. Let $X = (x, (x,t), t)$ be a point in $\mathcal{M}$ and let $\mathbf{b}$ be any vector in $\mathbf{R}^N$. Then

(1) $$\mathrm{v}(\mathcal{M}, X) - \mathrm{H}(\mathcal{M}, X) = \mathbf{b}$$

if and only if

(2) $$u_t - g^{ij}\,\mathrm{D}_{ij}u = \pi'\mathbf{b} - \mathrm{D}u(x,t)\circ\pi\mathbf{b},$$

where

$$\pi : \mathbf{R}^N \cong \mathbf{R}^m \times \mathbf{R}^{N-m} \to \mathbf{R}^m$$
$$\pi' : \mathbf{R}^N \cong \mathbf{R}^m \times \mathbf{R}^{N-m} \to \mathbf{R}^{N-m}$$

are the orthogonal projections, and where g^{ij} is the ij entry of the matrix whose inverse has ij entry equal to

$$g_{ij} = \delta_{ij} + \mathrm{D}_i u \cdot \mathrm{D}_j u.$$

Proof. Consider the linear map $\Pi = \pi' - \mathrm{D}u(x,t)\circ\pi$. Note that Π is the projection from $\mathbf{R}^m \times \mathbf{R}^{N-m}$ to $\mathbf{R}^{N-m}$ whose kernel is $\mathrm{Tan}(\mathcal{M}, X)$. Now

$$\begin{aligned} \mathrm{v} = \mathrm{v}(\mathcal{M}, X) &= \left(\frac{\partial}{\partial t}(x, u(x,t))\right)^{\perp} \\ &= (0, u_t)^{\perp} \\ &= (0, u_t) - (0, u_t)^{\tan}, \end{aligned}$$

so

(3) $$\Pi(\mathrm{v}) = u_t.$$

Similarly,

$$\begin{aligned} \mathrm{H} = \mathrm{H}(\mathcal{M}, X) &= \frac{1}{\sigma}\,\mathrm{D}_i\left(g^{ij}\sigma\,\mathrm{D}_j(x, u(x,t))\right) \\ &= g^{ij}\,\mathrm{D}_{ij}(x, u(x,t)) + \left(\frac{1}{\sigma}\,\mathrm{D}_i(g^{ij}\sigma)\right)\mathrm{D}_j(x, u(x,t)) \\ &= g^{ij}(0, \mathrm{D}_{ij}u) + \alpha^j\,\mathrm{D}_j(x, u(x,t)), \end{aligned} \tag{4}$$

where $\sigma = (\det g_{k\ell})^{1/2}$. The second term in (4) is a linear combination of vectors in $\mathrm{Tan}(\mathcal{M}, X)$. Thus

(5) $$\Pi(\mathrm{H}) = g^{ij}\,\mathrm{D}_{ij}u.t$$

If we apply the projection Π to both sides of (1), then (by (3) and (5)) we get (2). $\square$

If $N = m+1$, then g^{ij} simplifies to

$$\delta^{ij} - \frac{\mathrm{D}_i u\,\mathrm{D}_j u}{1 + |\,\mathrm{D}u|^2}.$$

For general codimensions, g^{ij} still depends smoothly on $\mathrm{D}u$ and is equal to δ^{ij} when $\mathrm{D}u = 0$. Thus we can rewrite (2) as

$$u_t - \Delta u = f(Du) + \pi'\mathbf{b} - Du\circ\pi\,\mathbf{b},$$

where $f(Du)$ is a smooth function of Du that vanishes when Du does.

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