SOLUTIONS TO THE SAMPLE EXAM PROBLEMS

1. Let $A$ be an $n \times n$ self-adjoint matrix. (That is, assume $A^* = A$.) Prove that all the eigenvalues of $A$ are real.

Solution: The fundamental property of the adjoint is:

$$(Au, v) = (u, A^*v).$$

Thus if $A^* = A$, then

$$(Au, v) = (u, Av).$$

Let $u = v$ be an eigenvector with eigenvalue $\lambda$. Then $(\dagger)$ becomes:

$$(\lambda v, v) = (v, \lambda v)$$

or

$$\lambda(v, v) = \overline{\lambda}(v, v)$$

or

$$\lambda |v|^2 = \overline{\lambda} |v|^2.$$ 

Since $v \neq 0$, this implies that $\lambda = \overline{\lambda}$, i.e., $\lambda$ is real. \(\square\)

2. Suppose $A$ is a self-adjoint $n \times n$ matrix. Suppose $u$ and $v$ are eigenvectors with different eigenvalues. Prove that $u$ and $v$ are perpendicular.

Solution Let $u$ have eigenvalue $\mu$ and $v$ have eigenvalue $\lambda$. Then $(\dagger)$ above becomes

$$(\mu u, v) = (u, \lambda v)$$

or

$$\mu(u, v) = \overline{\lambda}(u, v).$$

But the eigenvalues of a self-adjoint operator are real, so

$$\mu(u, v) = \lambda(u, v).$$

Since $\mu \neq \lambda$, this implies $(u, v) = 0$.

3. Prove (as in 2) that $u$ and $v$ are perpendicular, assuming $A$ is normal (i.e., $AA^* = A^*A$) but not necessarily self-adjoint.

Solution: Recall that, for normal operators, $Aw = \lambda w$ if and only if $A^*w = \overline{\lambda}w$. (See lemma 2 of the lecture notes on normal operators.)

Let $Av = \lambda v$ and $Au = \mu u$, $\lambda \neq \mu$. 

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Case 1: \( \lambda = 0 \). Then
\[
(u, v) = \frac{1}{\mu} (Au, v) = \frac{1}{\mu} (u, A^* v) = 0.
\]

Case 2: If \( \lambda \neq 0 \), apply case 1 to the normal operator \( B = A - \lambda I \).

4. Let \( V \) be a finite-dimensional complex vector space and \( T : V \to V \) be a linear operator. Suppose the only eigenvalue of \( T \) is 0. Prove that \( T^k = 0 \) for some \( k \).

**Solution 1:** We know \( V \) has a basis \( v_1, \ldots, v_n \) of generalized eigenvectors of \( T \). The only eigenvalue is 0, so for each \( i \), there is an \( m(i) \) so that
\[
(T - 0I)^{m(i)} v_i = 0.
\]
But then if we let \( m = \max\{m(1), \ldots, m(n)\} \),
\[
T^m (c_1 v_1 + \ldots + c_n v_n) = c_1 T^m v_1 + \cdots + c_n T^m v_n = 0.
\]
Thus \( T^m x = 0 \) for every \( x \), so \( T^m = 0 \).

**Solution 2:**

5. Let \( A \) be an \( n \times n \) real matrix such that \( Ax \) is nonzero and perpendicular to \( x \) for every nonzero \( x \). Prove that \( n \) is even.

**Solution:** We know that if \( n \) is odd, then \( A \) has a real eigenvalue \( \lambda \). Associated with \( \lambda \), there is a real eigenvector \( v \). Then
\[
0 = (Av) \cdot v = (\lambda v) \cdot v = \lambda |v|^2.
\]
Since \( v \neq 0 \), this means \( \lambda = 0 \), which means \( Av = 0 \), which contradicts the hypothesis. Thus \( n \) cannot be odd.

6. Suppose for a certain matrix \( A \) that
\[
A^2 - 3A + 2I = 0.
\]

(a) Prove that if \( \lambda \) is an eigenvalue of \( A \), then \( \lambda = 1 \) or \( \lambda = 2 \).

**Solution:** Let \( v \) be an eigenvector with eigenvalue \( \lambda \). Then
\[
0 = (A^2 - 3A + 2I)v = A(Av) - 3Av + 2v
= A(\lambda v) - 3\lambda v + 2v
= \lambda^2 v - 3\lambda v + 2v
= (\lambda^2 - 3\lambda + 2)v
= (\lambda - 2)(\lambda - 1)v.
\]
Thus (since $v \neq 0$) we must have $(\lambda - 2)(\lambda - 1) = 0$, which means $\lambda = 2$ or $\lambda = 1$.

(b)* Prove that either 1 or 2 must be an eigenvalue of $A$. (Both may be.)

**Solution:** Let $v$ be any nonzero vector. Then

$$0 = (A^2 - 3A + 2I)v = (A - 2I)(A - I)v.$$ 

Let $u = (A - I)v$. If $u = 0$, then $v$ is an eigenvector with eigenvalue 1. If $u \neq 0$, then $u$ is an eigenvector with eigenvalue 2 (since $(A - 2I)u = 0$).

NOTE: The hypotheses of the problem do NOT imply that 1 and 2 are both eigenvectors, for note that $A = I$ and $A = 2I$ both satisfy the equation $A^2 - 3A + 2I = 0$.

7. Suppose $T : V \to V$ is a linear operator on $d$-dimensional complex vector space $V$ and that $T$ has $d$ distinct eigenvalues $\lambda_1, \ldots, \lambda_d$. Prove that

$$T^n v \to 0 \quad \text{for all } v$$

if and only if $|\lambda_i| < 1$ for every $i$.

**Solution:** We know that when there are $d = \dim V$ distinct eigenvalues, then the corresponding eigenvectors $x_1, \ldots, x_d$ form a basis for $V$.

Any vector $v$ is a combination of the basis vectors:

$$v = c_1 x_1 + \cdots + c_d x_d,$$

so

$$T^n v = c_1 T^n x_1 + \cdots + c_d T^n x_d = c_1 \lambda_1^n x_1 + \cdots + c_d \lambda_d^n x_d.$$ 

Since each of these $d$ terms goes to 0 as $n \to \infty$, their sum also does.

8. Suppose $V$ is a finite dimensional vector space and that $W$ is a $k$-dimensional subspace. Prove that there is a basis for $V$ whose first $k$ elements are a basis for $W$.

**Solution:** Let $w_1, \ldots, w_k$ be a basis for $W$ and $v_1, \ldots, v_d$ be a basis for $V$. Then

$$w_1, \ldots, w_k, v_1, \ldots, v_d$$

spans $V$, so we know that we get a basis for $V$ by eliminating the vectors in this list that are linear combinations of preceding vectors. Since the $w$'s are independent, none of them gets eliminated. Thus we get a basis for $V$ consisting of all of the $w$'s plus some of the $v$'s.

9. Let $M$ be a $3 \times 3$ real symmetric matrix with eigenvectors $u = (1, 2, 3)$ and $v = (-1, -1, 1)$. Find a third eigenvector (not just a scalar multiple of $u$ or $v$.)
Solution: Since $M$ is symmetric (and real), $\mathbb{R}^3$ has an orthogonal basis of eigenvectors. We are given two, so the third must be perpendicular to those two. We can either get such a vector as $u \times v$, or we can solve

$$u \cdot x = 0, \quad v \cdot x = 0$$

or

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \end{bmatrix} x = 0.$$  

For instance, $(5, -4, 1)$ is such a vector.

10. Let $T$ be a $3 \times 3$ matrix all of whose entries are positive real numbers. Prove that $T$ has an eigenvector whose components are all nonnegative real numbers. (In fact the components will all be positive.)

Solution: Let $K$ be the triangle with corners at $e_1$, $e_2$, and $e_3$.

(In other words, $K$ is the set of $p = (p_1, p_2, p_3)$ such that each $p_i$ is $\geq 0$ and such that $p_1 + p_2 + p_3 = 1$.)

For $x \in K$, let $f(x)$ be the intersection of $K$ with the ray from 0 through $Tx$. (Note $Tx \neq 0$ for any $x \in K$, so this ray is well-defined.) Then $f : K \rightarrow K$ is continuous, so by Brouwer’s fixed point theorem, there is an $x \in K$ such that $f(x) = x$. But by definition, $f(x)$ is a scalar multiple of $Ax$.

11. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a bounded continuous map. (Bounded means there is an $M < \infty$ such that $|F(x)| \leq M$ for all $x$.) Prove there is an $x$ such that $F(x) = x$.

Solution: Let $B$ be the closed ball of radius $M$ centered at the origin. Note that $F$ maps $B$ to itself. Thus by Brouwer’s fixed point theorem, $F$ has a fixed point. □