Let $A : V \to V$ be an operator on the finite dimensional vector space $V$. What is the behavior of $A^n x$ as $n \to \infty$?

**Theorem 1.** Let

\[ Z = \{ x \in V : A^n x \to 0 \text{ as } n \to \infty \} \]
\[ Y = \{ x \in V : |A^n x| \text{ is bounded } n \to \infty \}. \]

Then $Z$ and $Y$ are subspaces of $V$ and $Z \subset Y \subset V$.

**Proof.** Exercise.

**Theorem 2.** Let $x$ be a generalized eigenvector of $A$ for the eigenvalue $\lambda \neq 0$. Let $k$ be the smallest number such that

\[ (A - \lambda I)^k x = 0. \]

Then

\[ \frac{A^n x}{\lambda^n n^{k-1}} \to u \]

as $n \to \infty$, where $u$ is an eigenvector with eigenvalue $\lambda$.

**Proof.** Let $N = (A - \lambda I)$. Then $A = \lambda I + N$. Since $\lambda I$ and $N$ commute, we can expand $A^n = (\lambda I + N)^n$ by the usual binomial formula:

\[ A^n = (\lambda I + N)^n = \sum_{j=0}^{n} \binom{n}{j} (\lambda I)^{n-j} N^j \]

so

\[ A^n x = \sum_{j=0}^{n} \binom{n}{j} (\lambda I)^{n-j} N^j x. \]

(*)
Since \( N^k \mathbf{x} = 0 \), all the terms of (*) with \( j \geq k \) vanish, so (for \( n \geq k \))

\[
A^n \mathbf{x} = \sum_{j=0}^{k-1} \binom{n}{j} \lambda^{n-j} N^j \mathbf{x}.
\]

Dividing by \( \lambda^n \) gives

\[
\frac{A^n \mathbf{x}}{\lambda^n} = \sum_{j=0}^{k-1} \binom{n}{j} \frac{N^j \mathbf{x}}{\lambda^j}.
\]

Note that

\[
\frac{1}{n^j} \binom{n}{j} = \frac{1}{n^j} \frac{(n-j+1) \cdots (n-1)n}{j!} = \frac{1}{j!} \left( 1 - \frac{j+1}{n} \right) \cdots \left( 1 - \frac{1}{n} \right) 1
\]

In this last expression, \( 1/(j!) \) is multiplied by \( j \) terms, each of which tends to 1 as \( n \to \infty \). Thus their product tends to 1 as \( n \to \infty \). That is,

\[
\lim_{n \to \infty} \frac{1}{n^j} \binom{n}{j} = \frac{1}{j!}.
\]

It follow that

\[
\lim_{n \to \infty} \frac{1}{n^p} \binom{n}{j} = 0 \quad \text{if } j < p.
\]

Thus if we divide both sides of (†) by \( n^{k-1} \) and let \( n \to \infty \), all but the last term (the \( j = k - 1 \) term) on the right will vanish:

\[
\lim_{n \to \infty} \frac{A^n \mathbf{x}}{\lambda^n n^{k-1}} = \frac{N^{k-1} \mathbf{x}}{j! \lambda^j}.
\]

Let \( \mathbf{u} \) be the vector on the right side of this equation. By choice of \( k \), \( \mathbf{u} \neq 0 \), but \( N \mathbf{u} = (A - \lambda I) \mathbf{u} = 0 \). That is, \( \mathbf{u} \) is a nonzero eigenvector of \( A \). \( \square \)

**Corollary.** Suppose \( A \) and \( \mathbf{x} \) are as in theorem 2. Then \( A^n \mathbf{x} \to 0 \) if and only if \(|\lambda| < 1\). Also, \( A^n \mathbf{x} \) stays bounded if and only if

\(|\lambda| < 1\), or \(|\lambda| = 1\) and \( k = 1\).

**Theorem 3.** Suppose \( A : V \to V \) is a linear operator on a finite-dimensional complex vector space. Let \( Z \) and \( Y \) be the subspaces defined in theorem 1. Then

1. \( Z \) is the subspace spanned by all generalized eigenvectors for eigenvalues \( \lambda_i \) with \(|\lambda_i| < 1\).
2. \( Y \) is the subspace spanned by all generalized eigenvectors for eigenvalues \( \lambda_i \) with \(|\lambda_i| < 1\) together with all eigenvectors with eigenvalues \( \lambda_i \) with \(|\lambda_i| = 1\).
Proof. We will only prove (1) as (2) is very similar. Let $W$ be the space spanned by all generalized eigenvectors for eigenvalues $\lambda_i$ with $|\lambda_i| < 1$. By the corollary to theorem 2, $Z$ contains $W$.

Note that $A$ maps $Z$ into itself. Thus there is a basis of $Z$ consisting of generalized eigenvectors of $A$. By the corollary to theorem 2, each of those eigenvectors must correspond to an eigenvalue $\lambda_i$ with $|\lambda_i| < 1$. Thus $Z \subset W$.

Since $W \subset Z$ and $Z \subset W$, in fact $Z = W$. □

**Corollary.** $A^n x \to 0$ for every $x \in V$ if and only if the eigenvalues of $A$ are all $< 1$ in absolute value.

Proof. We know that the generalized eigenvectors of $A$ span all of $V$. Thus if the eigenvalues are all $< 1$ in absolute value, then by theorem 3 the space $Z$ is all of $V$. That is, $A^n x \to 0$ for all $x$.

Conversely, if $|\lambda| \geq 1$ for some eigenvalue $\lambda$, then $A^n v \not\to 0$, where $v$ is an eigenvector with eigenvalue $\lambda$. □

**Real Vector Spaces**

Theorem 3 was stated and proved for complex vector spaces. But it has implications for real vector spaces.

**Theorem 4.** Let $A : V \to V$ be a linear operator on a finite-dimensional vector space $V$ (real or complex). Then $A^n x \to 0$ for all $x$ if and only if the eigenvalues (i.e., the roots of $\det(A - \lambda I)$) of $A$ are all $< 1$ in absolute value: $|\lambda_i| < 1$.

Proof. The corollary to theorem 3 already established this for complex spaces, so assume $V$ is a real vector space. Let $M$ be the matrix for $A$ (with respect to some basis.) Note that $A^n x \to 0$ for all $x \in V$ if and only if $M^n x \to 0$ for all $x \in \mathbb{R}^d$. (Here $d$ is the dimension of $V$.)

Let $z \in \mathbb{C}^d$ be a complex vector. Then $z$ can be written

$$z = x + iy$$

for suitable real vectors $x$ and $y$ in $\mathbb{R}^d$. Of course

$$M^n z = (M^n x) + i(M^n y).$$

From this we see that $M^n x \to 0$ for all real $x \in \mathbb{R}^d$ if and only if $M^n z \to 0$ for all complex $z \in \mathbb{C}^d$. But by the corollary to theorem 3, the latter happens if and only if the eigenvalues
of $M$ are all $< 1$ in absolute value. Of course $A$ and $M$ have the same eigenvalues, so we’re done. □

In the same way it is easy to prove that $A^n x$ stays bounded for all $x \in V$ if and only if:

1. the eigenvalues of $A$ are all $\leq 1$ in absolute value, and
2. for those eigenvalues $\lambda_i$ with $|\lambda_i| = 1$, every generalized eigenvector of $M : \mathbb{C}^d \to \mathbb{C}^d$ is an ordinary eigenvector.