Math 217a Homework 1

1. Let \( M \) be the space of lines \( L \) in \( \mathbb{R}^2 \). Consider the following two coordinate charts:

(a) \((u, v)\), where \( L \) is the line joining \((u(L), 0)\) and \((0, v(L))\). (b) \((r, w)\), where \( r(L) \) and \( w(L) \) are the polar coordinates for the point in \( L \) closest to the origin.

Express the vectorfields \( \frac{\partial}{\partial u} \) and \( \frac{\partial}{\partial w} \) in terms of the \((u, v)\) coordinate system.

2. Let \( V, W, \) and \( Z \) be the vectorfields on \( \mathbb{R}^2 \) such that \( \phi_tV \) is counterclockwise rotation by \( t \) about the origin and

\[
\phi_tV(x, y) = (x + t, y) \quad \phi_tZ(x, y) = (tx, ty)
\]

(a) Express all three vectorfields in terms of the basis \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \). (b) Find all three Lie brackets: \([V, W]\), \([V, Z]\), and \([W, Z]\).

3. Note that any linear mapping \( A : \mathbb{R}^n \to \mathbb{R}^n \) determines a tangent vectorfield \( \tilde{A} \) on \( \mathbb{R}^n \):

\[
\tilde{A}(p) = (p, Ap)
\]

Prove that if \( A, B : \mathbb{R}^n \to \mathbb{R}^n \) are linear maps, then

\[
[\tilde{A}, \tilde{B}] = \tilde{C}
\]

where \( C = AB - BA \).

4. Suppose \( Y_1, \ldots, Y_k \) are tangent vectorfields such that \([Y_i, Y_j] \equiv 0\) for all \( i \) and \( j \). Suppose \( p \) is a point such that \( Y_1(p), \ldots, Y_k(p) \) are linearly independent. Prove that there is a coordinate chart \((x^1, \ldots, x^n)\) defined on a neighborhood of \( p \) such that

\[
Y_i \equiv \frac{\partial}{\partial x^i} \quad (1 \leq i \leq k)
\]