THE APPLE GAME AND THE PRESSING-DOWN LEMMA

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Recall that $\omega_1$ is the smallest uncountable ordinal. In particular, $\omega_1$ is an uncountable well-ordered set, but for every $\alpha < \omega_1$ is countable.

The apple game is played in a series of stages, one for each element of $\omega_1$. At stage 0, you are given a countable collection of apples. At each stage $n \in \omega_1$ with $n > 0$:

1. If you have any apples, you must choose one and discard it.
2. If you have no apples, you get fined $1$.
3. In either case, you then are given a countably infinite new supply of apples.

Your goal is to have, at the end of the game, as many apples as possible and to have paid as few fines as possible.

It should be easy for you. After all, at each stage you discard at most 1 apple and you gain infinitely many apples. However:

**Apple Game Theorem.** After playing the apple game, you will have no apples and you will have paid uncountably many fines.

This theorem is also surprising for the following reason. Consider any stage $\alpha \in \omega_1$. Since $\alpha$ is countable, it can be put in one-to-one correspondence with the apples you received at stage 0. Thus you can arrive at time $a$ having paid no fines and having only used the apples you received at stage 0. In particular, you haven’t started to use any of the apples you were given at stages 1 through $a$. Nevertheless, according to the theorem, sooner or later you must run out of apples.

Here is yet another way to look at it. You can never pay a fine at any successor ordinal $\alpha + 1$, because you were just given apples at time $a$. So you can only have a problem at limit ordinals. Now every limit ordinal $L$ is followed by infinitely many successor ordinals (namely $L+1, L+2, \ldots$). Thus at “most” stages, it is impossible to run into trouble. Nevertheless, the trouble cannot be avoided.

**The Key Fact About $\omega_1$**

**Definition.** A subset $V$ of $\omega_1$ is called **bounded** if and only if there is an $\beta < \omega_1$ such that

$$\alpha \leq \beta \text{ for every } \alpha \in V.$$  (*

Note that (*) implies that $V \subseteq \gamma$ where $\gamma = \beta \cup \{\beta\} = \beta + 1$, which is $< \omega_1$ if $\beta < \omega_1$.  

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Theorem. A subset $V$ of $\omega_1$ is bounded if and only if it is countable.

Proof. If $V$ is bounded, then it is a subset of some $\gamma < \omega_1$. Since $\gamma$ is countable, so is $V$.

For the converse, recall that if $V$ is any set of ordinals, then $\beta = \bigcup V$ is the supremum or least upper bound of $V$. That is, $\beta$ is the smallest ordinal that is $\geq$ each element of $V$. Now if $V \subseteq \omega_1$, then $\beta = \bigcup V$ is a countable union of countable sets and is therefore countable, so $\beta < \omega_1$. □

Proof of the Apple Game Theorem

Suppose you have played the game. Let

$$A(y) = \{\text{the apples you received at stage } y \text{ or earlier}\}$$

$$= \bigcup_{x \leq y} \{\text{the apples you received at stage } x\}$$

Since $A(y)$ is a countable union of countable sets, it is countable. Let

$$T(y) = \{\tau < \omega_1: \text{at stage } \tau \text{ you discarded one of the apples in } A(y)\}.$$ 

Then $T(y)$ is a countable set of ordinals in $\omega_1$. (It is in one-one correspondence with a subset of $A(y)$.)

Thus $T(y)$ is bounded, so $T(y) \subseteq \alpha$ for some ordinal $\alpha < \omega_1$. Let $g(y)$ be the smallest such $\alpha$. In particular,

$$T(y) \subseteq g(y)$$

Note that $y + 1 \in T(y)$, so

(i) $y + 1 < g(y)$

Claim. For each time $x \in \omega_1$, there is a later time $b$ at which you were fined.

Proof. Define a sequence $x_n (n \in \omega)$ by

$$x_0 = x$$

$$x_{n+1} = g(x_n)$$

Since the set $X = \{x_n : n \in \omega\}$ is countable, it is bounded. Let $b$ be the least upper bound. That is, $b$ is the smallest ordinal such that

(ii) $x_n \leq b$ for all $n \in \omega$

From (i) we see that $x_0 < x_1 < \ldots$, so $x_i < b$ for all $i$. In particular, $x < b$.

I claim that you had no apple to discard at time $b$. For if you did discard an apple, you received that apple at some earlier time $t < b$. Now $t$ is NOT an upper bound for the $x_i$’s, so

$$t < x_n$$
for some \( n \in \omega \). That means that \( b \in T(x_n) \), which means that \( b < g(x_n) = x_{n+1} \), which contradicts (ii). This proves the claim.

But now the theorem follows immediately from the claim:

(1) According to the claim, the set of times at which you were fined is unbounded and therefore uncountable.
(2) Consider any apple that appeared in the game. It was given at some \( x \). But according to the lemma, there was a later time \( b \) at which you were fined. In particular, by time \( b \) you had already discarded the apple. Thus every apple was eventually discarded.

\[ \Box \]

Other Versions of the Theorem

One can restate the apple game theorem in purely mathematical language in several ways. Indeed, it is essentially a restatement of a theorem called the “pressing down lemma”.

Pressing Down Lemma (version 1). Consider a function

\[ T : \omega_1 \to \mathcal{P}(\omega_1) \]

that assigns to each \( y \in \omega_1 \) a countable subset \( T(y) \) of \( \omega_1 \). Let

\[ Z = \{ z \in \omega_1 : z \notin \bigcup_{y < z} V_y \} \]

Then \( Z \) is uncountable.

In terms of the game, \( T(y) \) is the set of times at which an apple received at time \( \leq y \) was discarded. Then \( Z \) is the set of times at which you were fined.

This theorem can be proved exactly as the apple game theorem was.

Pressing-Down Lemma (version 2). Let \( F : \omega_1 \to \omega_1 \) be a function such that

(1) \( F(x) \leq x \) for all \( x \in \omega_1 \), and
(2) For each \( y \in \omega_1 \), the set \( \{ x : F(x) = y \} \) is countable.

Then the set \( Z = \{ x : F(x) = x \} \) is uncountable.

This follows immediately from version 1: if we let \( T(y) \) be the set \( \{ x \in \omega_1 : F(x) \leq y \} \), then \( \{ x : F(x) = x \} \) is \( Z \).

Pressing-Down Lemma (version 3). Let \( F : \omega_1 \to \omega_1 \) be a function such that

\[ (*) \quad F(x) < x \text{ whenever } 0 < x < \omega_1. \]

Then for some \( y \in \omega_1 \), the set \( Z = \{ x : F(x) = y \} \) is uncountable.

This theorem says that a function \( F : \omega_1 \to \omega_1 \) that “presses down” (i.e., satisfies \((*)\)) is very far from being one-to-one. This is surprising because for any particular \( a \in \omega_1 \), you can make a pressing down function such that for each \( y \leq a \),

\[ \{ x : F(x) = y \} \]

has at most two elements.

Version 3 follows from version 2, because a counterexample to version 3 would (if we set \( F(0) = 0 \)) also be a counterexample to version 2.
The “real” pressing down lemma states somewhat more than the theorems above. It states that not only is the set $Z$ uncountable, but that its intersection with any closed unbounded subset $K$ of $\omega_1$ is also uncountable. (See Hrbacek-Jech, p. 208, for the definition of closed.) The proof is exactly as above, except that one defines $g(y)$ to be the least $\alpha \in K$ such that $T(y) \subseteq \alpha$.

A good example of a closed unbounded set is

$$K = \{ \delta^\mu : \mu < \omega_1 \}$$

where $\delta$ is some fixed countable ordinal (for example $\epsilon$.) In terms of the apple game, the “real” pressing down lemma says the following. Suppose I agree to enforce the fines only at those times $\tau$ in the set $K$. (In other words, you are not fined for running out of apples at other times.) Nevertheless, no matter what you do, you will still end up paying uncountably many fines.