MATH 121 HOMEWORK 8

1. For each prime number p, prove that there is a degree p polynomial in $\mathbb{Q}[x]$ whose Galois group is isomorphic to S_p.

2. Consider the polynomial: $x^5+2x^4+3x^3+4x+5$. Find the sum of the reciprocals of the roots.

3. Let K be a Galois extension of F. Suppose $[K : F]$ is divisible by p^m, where p is a prime number and $m \geq 1$.
(a). Prove that there is a field L such that $F \subset L \subset K$ and such that $[K : L] = p^m$.
(b). Suppose K is the splitting field of an irreducible polynomial $f(x) \in F[x]$. Let a be one of the roots of $f(x)$. Prove that the field L in part (a) may be chosen so that $a \notin L$.

4(a). Suppose $K \subset \mathbb{R}$ is a Galois extension of L such that $[K : L]$ is an odd prime number p. Suppose E and $E(r)$ are fields such that $[E(r) : E] = q$ for some prime number q, and such that

$$(*) \qquad K \cap E = L \neq K \cap E(r).$$

Prove that $q = p$ and that $E(r)$ is a Galois extension of E.
(b). Let K and L be as in part (a). Suppose E and $E(r)$ are fields such that (*) holds and such that $r^q \in E$ for some prime number q. Prove that $E(r) \not\subset \mathbb{R}$.
(c). Let K and L be as in part (a). Prove that if $E \subset \mathbb{R}$ is any real root extension of L, then $E \cap K = L$.
(d). Let F be a subfield of $\mathbb{R}$, and let $f(x) \in F[x]$ be an irreducible polynomial all of whose roots are real. Let K be the splitting field and suppose that $[K : F]$ is not a power of 2 (i.e., that $[K : F]$ is divisible by an odd prime p.) Let a be one of the roots of $f(x)$. Prove that a is not contained in any real root extension of F.

Remark: If K is the splitting field over F of an irreducible polynomial $f(x) \in F[x]$, then the degree of $f(x)$ divides $[K : F]$. Thus if the degree of $f(x)$ has an odd factor, then so does $[K : F]$, and thus the conclusion of (d) holds.

5. Express $\sum_{i \neq j} x_i^2 x_j$ as a polynomial in the elementary symmetric polynomials. (See exercises 37 and 38 in section 14.6 of the text.)

6. Suppose $f(x) \in \mathbb{Z}[x]$ is a monic polynomial of degree n with roots $\theta_1, \ldots, \theta_n$. Prove that there is a monic polynomial $g(x) \in \mathbb{Z}[x]$ of degree n whose roots are $\theta_1^2, \ldots, \theta_n^2$. (Indeed, given any $f(x)$, you can find the corresponding $g(x)$ even though you may not be able to find the roots of $f(x)$.)

7. (Not to turn in. For your enjoyment only.) Suppose F is a field with characteristic not equal to 2. Suppose K is a Galois extension of F such that $[K : F] = 2^n$ for some n. Prove that K is a root extension of F.