MATH 121 HOMEWORK 5

1. Determine the minimal polynomial over $\mathbb{Q}$ for $\sqrt{2}+\sqrt{5}$.

2. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a Galois extension of $\mathbb{Q}$ and that the Galois group is a cyclic group of order 4.

3. Let F be a field of characteristic $\neq 2$.
(a). Suppose D_1 and D_2 are elements of F such that none of D_1, D_2, and D_1D_2 have square roots in F. Let $K = F(\sqrt{D_1}, \sqrt{D_2})$. Prove that K/F is a Galois extension. Prove that the Galois group is the Klein 4-group.
(b). Conversely, suppose K/F is a Galois extension with Galois group isomorphic to the Klein 4-group. Prove that $K = F(\sqrt{D_1}, \sqrt{D_2})$ where D_1, D_2, and D_1D_2 are elements of F that do not have square roots in F.

4. Let K be the splitting field of $x^4 - 2x^2 - 2$ over $\mathbb{Q}$.
(a). Prove that $[K : \mathbb{Q}] = 8$.
(b). Prove that the Galois group is dihedral.

5. How many cube roots of 1 are there in $\mathbb{F}_{p^n}$? (The answer depends on p and n.)

6. Let F be a field and $F(t)$ be the field of rational functions.
(a). Let $p(t)/q(t)$ be an element of $F(t)$ that is not in F. Prove that $p(t)/q(t)$ is not algebraic over F. That is, prove that there is no monic polynomial $f(x) \in F[x]$ such that $f(p(t)/q(t)) = 0$.
(b). It follows from (a) that for each such element $p(t)/q(t)$, there is a unique endomorphism of $F(t)$ that fixes the elements of F and that takes t to $p(t)/q(t)$. [You can assume this, though you might want to figure out why it's true.] Show that the endomorphism is an isomorphism if and only if $p(t)/q(t)$ has the form $(at + b)/(ct + d)$ where $ad - bc \neq 0$.