MATH 121 HOMEWORK 2

1. Suppose that $[F(\alpha) : F]$ is odd. Prove that $F(\alpha) = F(\alpha^2)$.

2. Suppose that $F \subset R \subset E$ where F and E are fields and R is a ring. Suppose also that $\dim_F R < \infty$. Prove that R is a field.

3. Suppose $f(x) \in F[x]$ is an irreducible polynomial of degree n. Let $g(x)$ be any polynomial. Prove that every irreducible factor of $f(g(x))$ has degree divisible by n.

4(a). Suppose F is a field with three distinct roots of $x^3 - 1$. Let θ be an element of an extension field such that $\theta^3 \in F$. Prove that $[F(\theta) : F]$ must be 1 or 3.
(b) More generally, prove the same result assuming that some element (not necessarily 1) of F has three distinct cube roots in F.

5. Let $\theta_1, \ldots, \theta_n$ be elements of an extension field of F such that

$$(\theta_1)^3 \in F$$
$$(\theta_{i+1})^3 \in F(\theta_1, \ldots, \theta_i) \qquad (i < n).$$

(a). Prove that $[F(\theta_1, \ldots, \theta_n) : F]$ is either 3^k or $2 \cdot 3^k$ for some k.
(b). In the case $F = \mathbb{Q}$, prove that $2^{1/4} \notin F(\theta_1, \ldots, \theta_n)$.

6. Find $\left[\mathbb{Q}\left(\sqrt{3 + 2\sqrt{2}}\right) : \mathbb{Q}\right]$.

7. Suppose that F is a field with characteristic not equal to 2 and that D_1 and D_2 are elements of F, neither of which is a square in F.
(a). Prove that the degree of $F(\sqrt{D_1}, \sqrt{D_2})$ over F is 2 if $D_1 D_2$ is a square in F and 4 if not.
(b). If $D_1 D_2$ is not a square in F, prove that $[F(\sqrt{D_1} + \sqrt{D_2}) : F] = 4$ and that $F(\sqrt{D_1} + \sqrt{D_2}) = F(\sqrt{D_1}, \sqrt{D_2})$.