Problem 1 Determine which of the following subsets of $\mathbb{R}^2$ are open.

(a) $\{(x, y) \in \mathbb{R}^2 : (x - 3)^2 < 4 - (y - 5)^2\}$.

(b) $\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ or } y > 0\}$.

(c) $\{(x, y) \in \mathbb{R}^2 : 1 < x < 3 \text{ and } y = 0\}$.

How about $\{(x, y, z) \in \mathbb{R}^3 : e^x - y^2xz - z^3xy = 0\}$ in $\mathbb{R}^3$? Is it open or closed?

Problem 2

Let $f(x, y) = x^4 + y^4 - 2(x - y)^2$.

(a) Show that all the critical points of $f$ are $(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$.

(b) Use the Second Derivative Test to characterize each of $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ as a local minimum, local maximum, or neither.

(c) Classify $(0, 0)$ as a local minimum, local maximum, or neither, giving complete reasoning.

Problem 3

Find the maximum and minimum values of $f(x, y) = xy$ on the (closed and bounded) domain

$$D = \{(x, y) \in \mathbb{R}^2 | x^2 - xy + y^2 \leq 3\}.$$  

Show all of your reasoning (but you dont need to show that these extreme values exist).

Problem 4 Let $f(x, y) = e^{y-x+\sin x}$, and suppose $C$ is the curve in $\mathbb{R}^2$ defined by the equation $y^2 = x - x^4$.

(a) Show that the restriction of $f$ to $C$ must attain a maximum value.

(b) Show that the restriction of $f$ to $C$ does not attain its maximum value at the point $(0, 0)$.

Problem 5

The equation $8y^2 - 4x^3 + x^4 = 0$ defines a curve $C$ in $\mathbb{R}^2$, which is a closed, bounded set (you do not have to prove this). Notice also that the point $P = (3, 0)$ does not lie on $C$. Find both the shortest possible distance, and the longest possible distance, between $P$ and a point lying on the curve $C$; for each of these ‘extremal’ distances, list all points on $C$ that lie this distance from $P$. Show all steps in your reasoning.

Problem 6 (to do after Friday)

Find the minimum distance between the plane $x + 2y + z = \frac{25}{3}$ and the ellipsoid $x^2 + y^2 + 4z^2 = 1$. You can use without proof that such a minimum distance exists and that the plane and the ellipsoid do not intersect.