Section Worksheet 17
Solutions

Problem 1

a) \( \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x + 4xy - 6x^2y^2 \\ 4x^2y - 4x^3y \end{pmatrix} \)

\( \nabla f (1,1) = \begin{pmatrix} 2 + 4 - 6 \\ 4 - 4 \end{pmatrix} = \mathbf{0} \rightarrow (1,1) \) is a critical point.

b) \( Hf = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 + 4y^2 - 12xy^2 & 8xy - 12xy^2 \\ 8xy - 12xy^2 & 4x^2 - 4x^3 \end{pmatrix} \)

\( Hf(1,1) = \begin{pmatrix} -6 & -4 \\ -4 & 0 \end{pmatrix} \)

\( \det(Hf(1,1)) = -16 < 0 \rightarrow \) using the Second Derivative Test we conclude that \((1,1)\) is a saddle point. (We used that if \(A\) is \(2 \times 2\), \(\det A < 0 \rightarrow \lambda_1 < 0, \lambda_2 > 0\).

c) \( Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \) \( \rightarrow \) we can't use the Second Derivative Test. Note that \( f(x,y) = x^2(1 + 2y^2(1 - x)) \). Close to \((0,0)\) \( (1 - x) \approx 1 > 0 \rightarrow 2y^2(1 - x) > 0 \rightarrow 1 + 2y^2(1 - x) > 0 \Rightarrow x^2(1 + 2y^2(1 - x)) \geq 0 = f(0,0). \rightarrow (0,0) \) is a local min for \( f \).
Problem 2

a) \( D \) is closed and bounded \( \implies \) \( f \) has a global max on \( D \).

b) \( D \) is closed but not bounded \((x, y) \text{ can be arbitrarily large}\).

Note that \((0, 1) \in D \) for \( y \geq 2 \).

\[ f(0, 1) = y \rightarrow y \rightarrow \infty \implies f \text{ is unbounded} \]

\( \implies f \) has no global maximum.

c) \( D \) is closed but not bounded.

Note that if \((x, y) \in D \implies x \leq 0, y > 0 \)

\[ f(x, y) = x + y^2 \leq 0 - (+oo) = -oo \]

\( \implies x, y \leq 0 \)

\[ x \leq 0 \]

\[ -x^3 \leq 0 \]

\( \implies f(x, y) \leq 0 \rightarrow f(0, 4) \)

\[ f(0, 4) \in \text{a global maximum} \]

d) \( D \) is closed but not bounded

\( f(x, y, z) = x + y + 3z = x + y + z + y + z = 7 + y + 2z \)

The point \( \mathbf{g}(-2, 0, z+1) \in D \)

\[ f(-2, 0, z+1) = 1 + 0 + 2(z+1) = 2z + 3 \rightarrow oo \]

\( \implies f \) has no global max.

e) \( D \) is closed but not bounded, \( x, z \) can go to infinity only \( y \) and \( w \) are bounded.

\[ f(x, y, z, w) = x^2 + 1 \rightarrow oo \]

\( \implies f \) has no global max.
Problem 3

\[ \nabla f = (1, 1) + (0, 0) \rightarrow \text{no critical points in the interior of } D \]

Look on the boundary:

1. \( y = 0, x \in [-1, 1] \rightarrow f(x, 0) = x \rightarrow \frac{\partial f}{\partial x} = 1 \neq 0 \)
   \( \rightarrow \text{no critical points.} \)

2. \( x = 1, y \in (0, 1) \rightarrow f(1, y) = 2 + y \rightarrow \frac{\partial f}{\partial y} = 1 \neq 0 \)
   \( \rightarrow \text{no critical points.} \)

3. \( y = x^2, x \in [-1, 1] \rightarrow f(x, y) = x + x^2 \)
   \( \rightarrow \frac{\partial f}{\partial x} = 1 + 2x \)
   \( 1 + 2x = 0 \rightarrow x = -\frac{1}{2} \)
   \( -2y = \frac{1}{4} \)
   \( \rightarrow \left( -\frac{1}{2}, \frac{1}{4} \right) \)
   \( f\left( -\frac{1}{2}, \frac{1}{4} \right) = -\frac{1}{4} \)
(4) $x = -1$, $y \in [0,1]$  
\[ f(x, y) = y - 1 \]

\[ \frac{\partial f}{\partial y} = 1 \neq 0 \]

\[ \rightarrow \text{no cut points} \]

Now we look at the corners:

<table>
<thead>
<tr>
<th>$f(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1,0)$</td>
</tr>
<tr>
<td>$(1,1)$</td>
</tr>
<tr>
<td>$(1,1)$</td>
</tr>
</tbody>
</table>

\[ \rightarrow \text{The global min of } f \text{ is } -1 \text{ at } (1,0). \]
Problem 4

Look for critical points inside D:
\[
\nabla f = \begin{pmatrix} 2x - 4 \\ 4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad x = 2, \quad y = 0 \quad \text{but } (2, 0) \notin D
\]
because \(2^2 + 0^2 \neq 1\).

\(\rightarrow\) no critical points inside D.

Look at the boundary:

We can parameterize this part of the boundary by setting \(x = \cos t, \quad y = \sin t\).

\(-\)
\[
\phi(t) = \cos^2 t + 2 \sin t \cos t - 4 \cos t = 1 + 2 \sin^2 t - 4 \cos^2 t - 2 \sin t \cdot \cos t.
\]
\[
\phi'(t) = 2 \sin t \cos t + 4 \cos t \cdot (-\sin t) = 2 \sin t (\cos t + 2).
\]
\(\rightarrow\) \(2 \sin t = 0\) \(\Rightarrow t = 0\) or \(\cos t + 2 = 0\) impossible.

Because \(\cos t + 2 \geq 1 + 2 = 3 > 0\).
\( \min t = 0 \rightarrow \cos t = 0 \rightarrow \cos t = 1 \)

\[ f(t) = t^2 + 2 \cdot 0 - 4 \cdot 1 = 3 \]

at \((1,0)\).

2. we can parameterize \( \gamma \) by

\[ \begin{align*}
  x &= 0 \\
  y &= t
\end{align*} \]

\[ f(t) = 2t^2 \]

\[ f'(t) = 4t = 0 \]

\( \rightarrow t = 0 \)

\( \rightarrow \) look at \((0,0)\)

\[ f(0,0) = 0. \]

Now look at the corners:

\[ \begin{array}{c|c}
  x(0,1) &= 1 \\
  f(0,1) &= -1 \\
  \hline \\
  x(0,-1) &= 2 \\
  f(0,-1) &= -2 \\
\end{array} \]

\( \rightarrow \) global max of \( f \) on \( D \) is \( 2 \) at \((0,1)\), \((0,1)\), and \((0,-1)\).

\( \rightarrow \) global min of \( f \) on \( D \) is \(-3\) at \((1,0)\).
Problem 5

a) \[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{y}{x^2(y-2)} & \frac{\partial f}{\partial x} (3,1) &= 1 \\
\frac{\partial f}{\partial y} &= \frac{x}{x^2(y-2)} & \frac{\partial f}{\partial y} (3,1) &= 3 \\
\frac{\partial^2 f}{\partial x^2} &= \frac{-y^2}{(x^2(y-2)^2)} & \frac{\partial^2 f}{\partial x^2} (3,1) &= -1 \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{-2x}{(x^2(y-2)^2)} & \frac{\partial^2 f}{\partial x \partial y} (3,1) &= -2 \\
\frac{\partial^2 f}{\partial y^2} &= \frac{x^2}{(x^2(y-2)^2)} & \frac{\partial^2 f}{\partial y^2} (3,1) &= -9 \\
\end{align*}
\]

\[
\text{The Taylor polynomial is}
\]

\[
T_2(x,y) = 0 + 1(x-3) + 3(y-1) + \frac{(x-3)^2}{2} + 3 \left( \frac{y-1}{2} \right)
\]

\[
= (x-3) + 3(y-1) - \frac{(x-3)^2}{2} - 2(x-3)(y-1) - 9 \left( \frac{y-1}{2} \right)
\]

b) \[
T_2(2.9, 1.1) = \frac{\partial f}{\partial x} = -0.1 + 3 \cdot 0.1 + \frac{(0.1)^2}{2} - 2 \cdot 0.1 \cdot 0.1 \cdot 0.9 \cdot \frac{6.1}{2}
\]

\[
= 0.17
\]