Problem 1
Let $b$ be a scalar. Consider the matrix
$$A = \begin{bmatrix} 2 & b \\ b & 2 \end{bmatrix}$$
and the quadratic form $Q_A(x) = x^T Ax$.

(a) Find an orthonormal eigenbasis for $A$.

(b) Say the orthonormal eigenbasis you found in part (a) is $B = \{w_1, w_2\}$. Find an expression for $Q_A(uw_1 + vw_2)$. For which values of $b$ is $Q_A$ positive definite? How about indefinite? Explain in detail.

Problem 2
Consider the symmetric matrix
$$A = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$
and quadratic form $Q_A(v) = v^T Av$.

(a) For $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ give an explicit expression for $Q_A(v)$ in terms of $x, y, z$.

(b) Let $v_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. It is a fact that these are eigenvectors of $A$. Find each of the corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Determine the definiteness of the form $Q_A$.

(c) Let $w_i = \frac{v_i}{\|v_i\|}$. Then $B = \{w_1, w_2, w_3\}$ is an orthonormal eigenbasis of $A$. What is the expression for $Q_A$ in terms of $B$ coordinates. In other words give an explicit (non-matrix) formula for $Q_A(uw_1 + vw_2 + uw_3)$ in terms of $u_1, u_2, u_3$.

(d) Compute $Q_A(20v_1 + 10v_2 - 13v_3)$. Simplify as much as possible.

Problem 3
Let $f(x, y) = x^3 - 3xy^2 + 3y^2$.

(a) Find all critical points of $f$.

(b) For each critical point determine if it is a local maximum, a local minimum or a saddle point. Give a complete reasoning.

Problem 4
(a) For each of the functions $f_1(x, y) = x^4 + y^4$, $f_2(x, y) = x^4 - y^4$, $f_3(x, y) = -x^4 - y^4$, $f_4(x, y) = x^3 + y^3$ we know $(0, 0)$ is a critical point and we have $(Hf_i)(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. What can we conclude about the point $(0, 0)$ in each case using the Second Derivative Test? Can we say something about the point $(0, 0)$ in each case without using the Second Derivative Test?

(b) Let $g_1(x, y) = x^2 + y^4$, $g_2(x, y) = x^2 + y^5$. We know $(0, 0)$ is a critical point and $(Hg_i)(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. Same question as above.