Problem 1

Consider the matrix
\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}. \]

(a) Show that A has eigenvalues 0, 1, 4; and for each eigenvalue find a basis for the corresponding eigenspace. Show all reasoning.

(b) Does there exist a basis for \( \mathbb{R}^3 \) consisting of eigenvectors of A? If so, give one; if not, explain why not.

Problem 2

Suppose \( A \) is a symmetric 3 \( \times \) 3 matrix with eigenbasis \( \beta = \{v_1, v_2, v_3\} \) and associated eigenvalues \( \lambda_1 = 2, \lambda_2 = -1, \lambda_3 = 1. \)

(a) If \( x = c_1 v_1 + c_2 v_2 + c_3 v_3 \), use the information provided above to find an expression for \( A^5 x \).

(b) For this and part (c), suppose that two of the vectors of \( \beta \) are \( v_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix} \). Find, with reasoning, a valid possibility for the third vector \( v_3 \) which has unit length; simplify your answer as much as possible. (Hint: recall \( A \) is symmetric.)

(c) Find \( A^5 e_1 \), showing all reasoning; you may leave your answer expressed as an explicit linear combination of the vectors \( v_1, v_2, v_3 \) from part (b). (Hint: it may help to notice that \( v_1 \) and \( v_2 \) also have unit length.)

Problem 3

You are given the matrix \( A = \begin{bmatrix} a & b \\ b & 1/2 \end{bmatrix} \) for real numbers \( a, b \). We know that the determinant of \( A \) is 0 and that that \( \lambda_1 = 1 \) is an eigenvalue of \( A \).

(a) Explain why \( A \) cannot be the matrix of a rotation in \( \mathbb{R}^2 \).

(b) Determine the characteristic polynomial of \( A \) simplifying your answer as much as possible.

(c) Find, with reasoning, a pair of values \((a, b)\) for which all of the above conditions are true, and for this choice of \((a, b)\) give a simple verbal description of the linear transformation \( T(x) = Ax \).

Problem 4

Suppose \( \text{Proj}_L : \mathbb{R}^2 \to \mathbb{R}^2 \) is the linear transformation that projects vectors onto the line \( L \) spanned by \( \begin{bmatrix} 5 \\ 7 \end{bmatrix} \). Let \( A \) be the matrix of \( \text{Proj}_L \) with respect to the standard basis.

(a) Find, with justification, the two eigenvalues of \( A \).

(b) Find, with justification, a basis for each eigenspace of \( A \).

(c) Show that \( A^2 = A \).

Problem 5

For this problem, suppose \( A \) is an \( n \times n \) matrix.

(a) Complete the following sentence: A nonzero vector \( v \in \mathbb{R}^n \) is defined to be an eigenvector of \( A \) if

(b) Now suppose \( B \) is an invertible \( n \times n \) matrix, and let \( u \) be an eigenvector of \( B \) with eigenvalue \( b \). Show that \( u \) is an eigenvector of \( B^{-1} \), and find the corresponding eigenvalue.

(c) With \( A \) and \( B \) as above, suppose \( w \) is an eigenvector of the product \( AB \) with eigenvalue \( \lambda \). Show that \( Bw \) is an eigenvector of \( BA \), and find the corresponding eigenvalue.