Section Worksheet 11

Solutions

Problem 1

1. We use the chain rule:

\[ Df = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} y + 2 & x + 2 & y + x \end{pmatrix} \]

\[ Dg = \begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_2}{\partial u} & \frac{\partial g_3}{\partial u} \\ \frac{\partial g_1}{\partial v} & \frac{\partial g_2}{\partial v} & \frac{\partial g_3}{\partial v} \\ \frac{\partial g_1}{\partial w} & \frac{\partial g_2}{\partial w} & \frac{\partial g_3}{\partial w} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\sin u & 0 \\ 0 & 0 & -\sin u \end{pmatrix} \]

\[ D(g \circ f) = (Dg)(f(x)) \cdot (Df)(x) = \begin{pmatrix} 2(x + y + 2z + x) & y + 2 & x + 2 \end{pmatrix} \cdot \begin{pmatrix} y + x \\ \sin(y + y + 2z + x) \\ -\sin(y + y + 2z + x) \end{pmatrix} \]

\[ = \begin{pmatrix} 2(x + y + 2z + x)(y + x) & 2(x + y + 2z + x)(y + x) & 2(x + y + 2z + x)(y + x) \\ -\sin(y + y + 2z + x)(y + x) & -\sin(y + y + 2z + x)(y + x) & -\sin(y + y + 2z + x)(y + x) \end{pmatrix} \]
1. The second point implies that the gradient \( \nabla f \) at \((2,1)\) is non-zero and parallel to \((\frac{3}{5}, \frac{4}{5})\). The gradient is perpendicular to the level set \(C\), hence the gradient is perpendicular to the tangent line to \(C\) at \((2,1)\). The tangent line to \(C\) at \((2,1)\) will then be parallel to a vector \((a, b)\) such that \((a, b) \cdot (\frac{3}{5}, \frac{4}{5}) = 0\). We can choose \((a, b) = (\frac{4}{5}, -\frac{3}{5})\). The slope is then \(-\frac{3}{5}/\frac{4}{5} = -\frac{3}{4}\).
2. The tangent plane to \( S \) at \( x=2, y=1 \) has the equation \( z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \).

We know \( f(2,1) \), we need to find \( f_x(2,1), f_y(2,1) \), i.e., we need to find \( \nabla f(2,1) \).

By the second point, \( \nabla f(2,1) \) is parallel to \( \left( \frac{3}{5}, \frac{4}{5} \right) \)

\[ \nabla f(2,1) = c \cdot \left( \frac{3}{5}, \frac{4}{5} \right) \]

By the third point, second point \( D_u(2,1) = \nabla f \cdot \mathbf{u} = 5 \)

\[ \mathbf{u} = \left( \frac{3}{5}, \frac{4}{5} \right) \]

\[ c \cdot \left( \frac{3}{5}, \frac{4}{5} \right) = 5 \quad \Rightarrow \quad c = \frac{5}{\frac{3}{5} + \frac{4}{5}} = 5 \quad \Rightarrow \quad c = \frac{9 + 16}{25} = \frac{25}{25} = 1 \]

\[ f_x(2,1) = \frac{3}{5} \quad \Rightarrow \quad f_y(2,1) = \frac{4}{5} \]

\[ \Rightarrow \quad \nabla f(2,1) = \left( \frac{3}{5}, \frac{4}{5} \right) = \left( 3, 4 \right) \]

\[ f_x(2,1) = 3 \quad \Rightarrow \quad f_y(2,1) = 4 \]

\[ \nabla f(2,1) = 1 \cdot \left( \frac{3}{5}, \frac{4}{5} \right) \]

\[ \Rightarrow \quad \text{the eq. of the tangent plane is} \]

\[ z = 8 + 3(x-2) + 4(y-1) \]

\[ \text{Note: We did NOT use the third point to solve this problem. This is It was not needed.} \]
Problem 3

The direction of the most rapid increase is the gradient vector $\nabla f$ and the direction of the most rapid decrease is $-\nabla f$.

1. The direction of most rapid decrease in $p$ at $(1,0,0)$ is $-\nabla p 
\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ first row in $Dm^2$ transposed.

we need the unit direction which is

$$\frac{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{1^2+0^2+1^2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

2. We need to find the directional derivative of $s$ in the direction $\vec{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

$D_{\vec{v}} s(1,0,0) = \nabla s(1,0,0) \cdot \vec{v} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = -\frac{4}{\sqrt{2}}$

The rate of change of $s$ in the $\vec{v}$ direction is neg.

$\rightarrow \ s \ \text{will decrease.}$
3. Reformulating the question: Can we find a direction \( \tilde{w} \) such that
\[
\begin{align*}
\nabla w(1,0,0) \cdot \tilde{w} &= 0 \quad (\text{w stays constant in } \tilde{w} \text{ direction}) \\
\nabla P(1,0,0) \cdot \tilde{w} &\leq 0 \quad (P \text{ decreases in the } \tilde{w} \text{ direction})
\end{align*}
\]

(=)

\[
\begin{cases}
(0) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0 \\
(1) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \leq 0
\end{cases}
\]

\begin{align*}
&\begin{pmatrix} w_1+w_3 = 0 \\
&\begin{pmatrix} w_1+w_3 < 0
\end{pmatrix}
\end{align*}

This system has solutions, for example,
\[
\tilde{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}
\]

Problem 4

The direction of most rapid decrease is \(-\nabla f\).

\[
\nabla f(1,2) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}(1,2) = \begin{pmatrix} -1 \\ -x+2y \end{pmatrix}(1,2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1,2) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}
\]

\[
= \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}
\]
The equation of the tangent plane is
\[ 0 = F_x(0, -2, 1)(x - 0) + F_y(0, -2, 1)(y + 2) + F_z(0, -2, 1)(z - 1) \]

\[ F_x = 5, \quad F_x(0, -2, 1) = 0 \]
\[ F_y = -3, \quad F_y(0, -2, 1) = 12 \]
\[ F_z = -6, \quad F_z(0, -2, 1) = -12 \]

\[ \Rightarrow \] the equation of the tangent plane is
\[ 12(x + 2) + 12(y + 2) = 0 \]
\[ 12y - 12z = -36. \]