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Mathieu-12 challenge
November 26, 2010

A Straightforward Solution to the Topsy Turvy Puzzle

Let L and R be the elements of S_12 that map the vector (1, 2, …, 12) to (11, 9, 7, 5, 3, 1, 2, 4, 6, 8, 10, 12) and (2, 4, 6, 8, 10, 12, 11, 9, 7, 5, 3, 1) respectively, as in the Topsy Turvy puzzle, and let M_12 be the subgroup of S_12 generated by L and R. We intend to find a solution for the puzzle; that is, given a permutation of {1, …, 12}, we will construct a finite product of R's and L's that yields the identity permutation when applied to the given permutation—if such a product exists. Throughout this solution, XY will refer to the process of applying Y, then X.

Our general strategy will be to solve pieces 1, 2, and so on, until eventually (and surprisingly soon) the entire puzzle is forced to be solved. For each step, we will use two basic tools to construct move sequences that preserve our progress. First, when we already have a move sequence with one or more fixed points, we can conjugate that sequence to fix equally many points of our choosing instead. Second, when we have no other resources available (especially later in the solution) we will apply a somewhat arbitrary "scrambling sequence" to the puzzle, and then solve the puzzle as far as we can from the resulting position. If the scrambling sequence is chosen correctly, the combined scrambling and partially solving sequence will then give us a permutation we were previously unable to reach.

It is a consequence of the recursive nature of both tools that the sequences found late in the solution are relatively long. While we aim for efficiency at every step, inefficiency is the inevitable price paid for the low-tech nature of the solution—nothing more than Mathematica's ability to multiply permutation matrices rapidly was exploited in deriving this method, and computers are not needed to apply it. The use of computer searches would certainly enable us to cut the total move count; since M_12 has only 95,040 elements, a brute force search would be an easy calculation by computer.

Before starting our solution, we note that L has the cycle notation (1 6 9 2 7 3 5 4 8 10 11), an 11-cycle, and R is (1 12 6 3 11 7 9 8 4 2)(5 10), a 10-cycle and a transposition. Since L^-1 = L^10 and R^-1 = R^9, any sequence of L's and R's can be inverted move-by-move. It will often be more convenient, however, to invert a sequence as a whole; that is, if X is a group element with order n, then we have X^(n-1) = X^-1.

Step 1: solve piece 1.
First suppose piece 1 is not in position 12. Then, since L cycles all pieces except piece 12, repeated application of L will bring piece 1 to its solved position. If piece 1 is in position 12, then LRRR will solve piece 1.

Step 2: solve piece 2.
We would like a move sequence that fixes piece 1 and cycles all other pieces, thereby solving piece 2 from any position upon repetition. We observe that L cycles all pieces except piece 12,

and R maps piece 1 to piece 12. So the conjugate (R^-1)LR = (R^9)LR, which we denote as A, cycles all pieces but 1. Repeated application of this will solve piece 2. Note that since A is a conjugate, the conjugating moves cancel when powers of A are expanded, so we can write A^n simply as (R^9)(L^n)R.

Step 3: solve piece 3.
Here we need some more sophisticated move sequences. First, recall that sequence A above fixes piece 1. We can generate several other sequences that fix piece 1 by applying R a number of times, then moving piece 1 back to its original position as in Step 1. Two of these sequences are L(R^4) and (L^3)(R^7). From each of these, we can generate a sequence that fixes the first two pieces by solving piece 2 as in Step 2. The resulting sequences are B = (A^2)L(R^4) and C = A(L^3)(R^7), which have cycle notations (3 9 8 4)(5 6 7 11) and (3 12 6)(4 11 7)(5 10 8), respectively.

Now, suppose pieces 1 and 2 are in their solved positions and piece 3 is in any other position. If piece 3 is in position 10, then applying (B^2)C will solve it. If it is in position 12, then C^2 will solve it. If it is anywhere else, then up to three applications of B will move it to position 3 or position 6, and then C will move it from 6 to 3 if necessary. This solves piece 3.

Step 4: solve piece 4.
We now build on the sequences B and C to solve a fourth piece. We will need two sequences here. First, since C fixes piece 9 and B moves piece 3 to position 9, we have the conjugate D = (B^-1)CB = (B^3)CB, with cycle notation (4 12 5)(6 8 7)(9 11 10), which fixes the first three pieces. Then, using the somewhat arbitrary scramble sequence B(C^2)(B^3), we construct the sequence E = B(C^2)B(C^2)B^3, with cycle notation (4 6 11)(5 7 9)(8 10 12), which also fixes pieces 1, 2, and 3. If the first three pieces are solved, then applying E zero, one, or two times will move piece 4 to position 4, 5, or 12, and then applying D zero, one, or two times will solve it.

Step 5: solve piece 5.
Now we construct a move sequence that fixes pieces 1, 2, 3, and 4, but not the other pieces. Recall that our sequence B fixes pieces 1, 2, 10, and 12, and permutes the other eight pieces in two four-cycles. We will construct our next sequence by conjugating B. To do this, we must find a sequence which maps the set {1, 2, 10, 12} to {1, 2, 3, 4}. Steps 3 and 4 allow us to do just that: we move piece 12 to position 3 by applying C^2, then move piece 10 (which is now in position 5) to position 4 by applying D. So the conjugate D(C^2)B(C^-2)(D^-1) fixes pieces 1, 2, 3, and 4, and it moves the rest in two 4-cycles. Since both C and D have order 3, we simplify this as D(C^2)BC(D^2); this sequence has cycle notation (5 10 12 7)(6 8 11 9), and we label it F.

This last sequence allows us to solve piece 5 from positions 5, 7, 10, or 12, but we do not yet have a way to solve it from positions 6, 8, 9, and 11. To accomplish this, we revert to the method of scrambling and resolving. If we apply the scrambling sequence R^9, then we can solve piece 1 with L^8, piece 2 with A^9, piece 3 with C(B^2), and piece 4 with E. Combined, this gives us our last sequence, EC(B^2)(A^9)(L^8)(R^9), which has cycle notation (5 8 12 9)(6 7 11 10), and which we denote G. Then, if piece 5 is in position 6, 8, 9, or 11, we apply G once

to move it (respectively) to position 7, 12, 5, or 10; and we can then solve piece 5 using up to three applications of F.

These steps are sufficient to solve pieces 1, 2, 3, 4, and 5 from any positions in the puzzle. But since M_12 is sharply 5-transitive, any element of M_12 with pieces 1 through 5 solved has all pieces solved. Therefore, the steps given above will solve the Topsy Turvy puzzle if and only if it begins in a solvable position.

For reference, we list below the full expansions of all sequences constructed above. Recall that since the sequences A, D, and F were constructed as conjugates, the conjugating moves can be cancelled when expanding powers of them. We simplify the last three sequences further by canceling the identity R^10 = 1. All sequences are to be executed right-to-left.

A = (R^9)LR = RRRRRRRRRLR (11 moves; powers up to A^10 will take up to 20 moves by cancellation)

B = (A^2)L(R^4) = RRRRRRRRRLLRLRRRR (17 moves)

C = A(L^3)(R^7) = RRRRRRRRRLRLLLRRRRRRR (21 moves)

D = (B^3)CB = RRRRRRRRRLLRLRRRR RRRRRRRRRLLRLRRRR RRRRRRRRRLLRLRRRR RRRRRRRRRLRLLLRRRRRRR RRRRRRRRRLLRLRRRR = RRRRRRRRRLLRLRRRLLRLRRRLLRLRRRLRLLLRRRRRRLLRLRRRR (49 moves; D^2 cancels down to RRRRRRRRRLLRLRRRLLRLRRRLLRLRRRLRLLLRRRRRRLRLLL RRRRRRLLRLRRRR, which is 60 moves)

E = B(C^2)B(C^2)B^3 = RRRRRRRRRLLRLRRRR RRRRRRRRRLRLLLRRRRRRR RRRRRRRRRLRLLLRRRRRRR RRRRRRRRRLLRLRRRR RRRRRRRRRLRLLLRRRRRRR RRRRRRRRRLRLLLRRRRRRR RRRRRRRRRLLRLRRRR RRRRRRRRRLLRLRRRR RRRRRRRRRLLRLRRRR = RRRRRRRRRLLRLRRRLRLLLRRRRRRLRLLLRRRRRRLLRLRRRLRLLLRRRRRRLR LLLRRRRRRLLRLRRRLLRLRRRLLRLRRRR (89 moves)

F = D(C^2)BC(D^2) = RRRRRRRRRLLRLRRRLLRLRRRLLRLRRRLRLLLRRRRRRLLRLRRRR RRRRRRRRRLRLLLRRRRRRR RRRRRRRRRLRLLLRRRRRRR RRRRRRRRRLLRLRRRR RRRRRRRRRLRLLLRRRRRRR RRRRRRRRRLLRLRRRLLRLRRRLLRLRRRLRLLLRRRRRRLLRLLLRRRRRRLLRLRRRR = RRRRRRRRRLLRLRRRLLRLRRRLLRLRRRLRLLLRRRRRRLLRLRRRLRLLLRRRRR RLRLLLRRRRRRLLRLRRRLRLLLRRRRRRLLRLRRRLLRLRRRLLRLRRRLRLLLRRRR RRLRLLLRRRRRRLLRLRRR (139 moves; F^2 cancels down to 146 moves and F^3 is 153)

G = EC(B^2)(A^9)(L^8)(R^9) = RRRRRRRRRLLRLRRRLRLLLRRRRRRLRLLLRRRRRRLLRLRRRLRLLLRRRRRRLRLL LRRRRRRLLRLRRRLLRLRRRLLRLRRRR RRRRRRRRRLRLLLRRRRRRR

RRRRRRRRRLLRLRRRR RRRRRRRRRLLRLRRRR RRRRRRRRRLLLLLLLLLR
LLLLLLLL RRRRRRRRR
= RRRRRRRRRLLRLRRRLRLLLRRRRRRLRLLLRRRRRRLLRLRRRLRLLLRRRRRRLR
LLLRRRRRRLLRLRRRLLRLRRRLLRLRRRLRLLLRRRRRRLLRLRRRLLRLRRRLLLLL
LLLLRLLLLLLLLRRRRRRRRR (140 moves)

Combining worst cases, we have a theoretical maximum of 10 + 20 + (17*3 + 21) + (60 + 89*2) + (153 + 140) = 10 + 20 + 72 + 238 + 293 = 633 moves to solve the puzzle.