Solving Topsy-Turvy

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1 Introduction

The numbers 1,2,...,12 are initially in order. They are scrambled using some sequence of two permutations known as Left and Right. Left takes $1-2-3-4-5-6-7-8-9-10-11-12$ to $11-9-7-5-3-1-2-4-6-8-10-12$, and Right takes $1-2-3-4-5-6-7-8-9-10-11-12$ to $2-4-6-8-10-12-11-9-7-5-3-1$. We are then given the scrambled numbers, and our task is to unscramble them using the same two permutations. How do we do it?

The subgroup of the symmetric group S_{12} generated by Left and Right is known as M_{12}, where M is for Mathieu. M_{12}, of order $95040 = 12!/7!$, is the second-smallest sporadic simple group. The order of M_{12} arises from the fact that it is sharply 5-transitive, meaning that given any map from some 5-element subset A of $\{1, 2, ..., 12\}$ to some other 5-element subset B of $\{1, 2, ..., 12\}$, there is exactly one permutation in M_{12} whose action on $\{1, 2, ..., 12\}$ restricts to that map on A. In particular, the identity is the only element of M_{12} whose last five elements are $8, 9, 10, 11, 12$ in that order. So if we manage to get $8, 9, 10, 11, 12$ in the right place by applying Left and Right repeatedly, either the rest is already solved or the original scrambling we were given was unsolvable. Now we just have to put each of these numbers in place one by one.

2 Placing 12 and 11

Getting the last two numbers right is extremely easy. In cycle notation, we have

$$\text{Left } = (1-6-9-2-7-3-5-4-8-10-11)(12)$$

$$\text{Right } = (1-12-6-3-11-7-9-8-4-2)(5-10)$$

We can see that in most cases, we can simply apply Right over and over again until 12 is cycled into the right place. This will only fail to work if 12 starts out in the 5th or 10th slot, in which case it suffices to apply Left once before cycling the 12 in using Right. Once the 12 is in, we can use Left repeatedly until 11 cycles into the right place, since Left has no effect on the last number.

3 Placing 10, 9, and 8

This stage of solving is rather less intuitive. For now, we simply list 24 algorithms to place the 10, 9, and 8 one by one. It is probable that experimentation would reveal shortcuts or methods easier to remember. However, the current list is not an unbearable amount of information for a dedicated human to memorize, and is reasonable as a starting point.

3.1 Placing the 10

The number on the left is the starting position of the 10, with 11 and 12 already in place. The sequence given leaves all of 10, 11, and 12 in the correct positions.

9 LRRRLRL
8 RLLRLLLRR
7 RLRLRLRRR
6 LLRLRRRRRL
5 LRRLRRRLLR
4 LRRRLLLLRRL
3 LLRRLLLLLLR
2 RLRLLLRLL
1 RLLRRLLLL

3.2 Placing the 9

The number on the left is the starting position of the 9, with 10, 11, and 12 already in place. The sequence given leaves all of 9, 10, 11, and 12 in the correct positions.

8 LLRRLRLLLRRLRL
7 RLLLLRRRLRR
6 RRLRRLLRLLLLR
5 LLLRLRLLLRLLLL
4 LLLLLRLLRRLL
3 RLRLRLLRLLLLR
2 RRRRLRRRLRR
1 LLRRRLR

3.3 Placing the 8

The number on the left is the starting position of the 8, with 9, 10, 11, and 12 already in place. The sequence given leaves all of 8, 9, 10, 11, and 12 in the correct positions, and therefore finishes the puzzle.

7 RRRRLRLLRLRLLR
6 LRLLRLLLRLLRLLLR
5 LLRLLRLRLRRRRLRRRL
4 LLRRRRRRLRRRLLLLLR
3 LRLLLLRLRLL
2 RLLLLRLLRLRLLR
1 RLLLRRRRRR

4 Acknowledgements

I would like to thank David Fisher for teaching me the programming necessary to complete the tables above.

5 References

O. van Deventer. Developing Topsy-Turvy and Number Planet. Available at `http://oskarvandeventer.nl/M12/Developing_Topsy_Turvy_and_Number_Planet.doc.pdf` (2010/22/11).