Solution to the Mathieu-12 Puzzle

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1 Notation

For the rest of the section, twelve numbers enclosed by parentheses denote a state of the Mathieu-12 Puzzle. It doesn't represent a cycle decomposition of the permutation; rather, it shows all the numbers and their locations from the left to the right. There are L and R moves which can be applied to the puzzle, and the characters L and R are concatenated together to denote a series of moves. For example, LR means an L move followed by an R move. Also, L^k is a shorthand for k L moves, and R^k is defined similarly. A dot $(\cdot)$ denotes the series of zero moves; that is, no moves.

2 Introduction

The Mathieu-12 Puzzle has twelve tokens, which can be permuted using two different moves:

- Move L sends (1 2 3 4 5 6 7 8 9 10 11 12) to (11 9 7 5 3 1 2 4 6 8 10 12).
- Move R sends (1 2 3 4 5 6 7 8 9 10 11 12) to (2 4 6 8 10 12 11 9 7 5 3 1).

Given a permutation of twelve tokens, the goal of the puzzle is to rearrange them into other. It'll be shown that the set of reachable states form a subgroup M_{12} of S_{12} with index 5040. This gives $|M_{12}| = 95040$, which makes it easy to enumerate all reachable states by an elementary graph search algorithm.

3 Properties

With breadth-first-search (BFS), the following facts could be obtained:

- The set of actions $\{L, R\}$ yields 5040 different orbits in S_{12}, where the size of each orbit is 95040. Within each orbit, each number appears 7920 times at each of twelve different positions.

- Let s_k be the number of states that require minimum of k moves. The following table gives the values of s_k for all k.

k	0	1	2	3	4	5	6	7
s_k	1	2	4	8	16	32	64	128
k	8	9	10	11	12	13	14	15
s_k	256	512	1023	1950	3724	6880	12248	19448
k	16	17	18	19	20	21		
s_k	22592	16448	7072	1976	480	136		

  The table shows that there are no relations between moves of length less than 10, and there exists only one relation of length 10, namely $R^{10} = 1$.

- The only rotations of (1 2 3 4 5 6 7 8 9 10 11 12) that can be obtained by L and R moves are itself and (7 8 9 10 11 12 1 2 3 4 5 6). The latter can be obtained by applying R^9L.

- For a solvable puzzle, the locations of five different tokens completely decide the locations of other tokens. In particular, if five tokens are in place, all other tokens have to be in place too. Otherwise, the puzzle is unsolvable. Conversely, any set of five tokens can occupy any set of five (ordered) different locations.

4 Solving the Mathieu-12 Puzzle

We describe the complete solution to the Mathieu-12 Puzzle in this section. Using the property above, we will proceed by putting five tokens in the right place, one by one. The method described here is not at all the most memorable one, but it provides a systematic way to obtain any reachable state in a reasonable number of moves. The algorithm works as the following:

1. Bring 3 to the right place.
2. Bring 4 to the right place without moving 3 to somewhere else.
3. Bring 6 to the right place without moving 3 or 4 to somewhere else.
4. Bring 5 to the right place without moving 3, 4 or 6 to somewhere else.
5. Exchange the first six and the last six tokens to bring 3, 4, 5, and 6 to the right end of the sequence.
6. Bring 2 right in front of 3.
7. Exchange the first six and the last six to solve the puzzle.

Step 5 and 7 can be done by applying R^9L, as described above. Now, the following tables describe which move(s) should be applied in order to complete each step. Let p_3 be the location of token 3 in the beginning of step 1. Applying the move(s) in the following table brings token 3 to the right place, finishing step 1.

p_3	Move(s)
1	LR
2	L^2
3	$\cdot$
4	RL^2
5	LRL^2
6	R
7	L
8	R^2L^2
9	L^3
10	LRL
11	RL
12	R^2

Other steps can be done similarly. Define p_4 be the location of token 4 in the beginning of step 2, and define p_6 and p_5 likewise. Then, use the following table to solve the puzzle.

p_4	Move(s)	p_6	Move(s)	p_5	Move(s)
1	LRL^3R	1	$RLR^2L^2R^4$	1	$L^2R^3LRL^4$
2	LRL^2R^3	2	LRL^2R^5L	2	$(RL^2)^3$
3	$\times$	3	$\times$	3	$\times$
4	$\cdot$	4	$\times$	4	$\times$
5	R^2L	5	$L^2(RL)^3L^4$	5	$\cdot$
6	$(LR)^2L$	6	$\cdot$	6	$\times$
7	L^5RL	7	RLR^3	7	RL^6RL^2R
8	RL^2R^3L	8	$R^2L^4R^3LR$	8	$R^2LRL^3RL^2RL$
9	$(LR)^3RL$	9	$R^4L^3R^3$	9	$(LR)^4RL^4$
10	RL^2R	10	$LR^3(LR)^2RL$	10	$L^4R^2(LR)^2L^2R$
11	L^2RL^2	11	$R^2L^2(RL)^2$	11	$(L^2R^2)^2RL^2$
12	$(R^2L)^2$	12	$R^2LR^3L^3$	12	$L^3R^3LR^3L$

Step 6 can be simplified by noting that all possible configurations of the remaining 8 tokens form a group isomorphic to the quarternion group Q_8. Specifically, the sequence of moves $R^4(LRL^2R)^2$ corresponds to i, $LR(L^2RL^3R)^2$ corresponds to j, $L^2R^6LR^3L^5R$ corresponds to k, and RL^3R^6 correponds to -1. Let I, J, K, -1 be the series of moves described above. The following table shows how these moves permute the sequence.

Move	Result
$\cdot$	(1 2 3 4 5 6 7 8 9 10 11 12)
I	(2 8 4 6 3 5 1 7 9 10 11 12)
J	(3 5 8 2 7 1 4 6 9 10 11 12)
K	(5 6 2 1 8 7 3 4 9 10 11 12)
-1	(8 7 6 5 4 3 2 1 9 10 11 12)

One can see that applying J followed by I is equivalent to applying K, and etc. Thus, some combination of these moves is enough to finish the last step of the puzzle.

5 Conclusion

The Mathieu-12 Puzzle can completely be solved by brute-force, thanks to the small size of the problem. However, using the structure of the problem, it is possible to construct a systematic way to solve a puzzle given any (solvable) permutation. The similarity between this algorithm and the solution to Rubik's cubes is notable; the algorithm starts with a completely unstructured state but puts the tokens in place, one by one. As more tokens are placed in the right position, the complexity of next moves drastically increase, since it becomes harder and harder to move other tokens without spoiling the tokens that are already in the right place.

The method described in this report is not at all the easiest way to solve the puzzle. For example, Steps 5 and 7 could have been omitted if we started by bringing tokens from 9 to 12 to the right locations, instead of tokens from 3 to 6. The reason for this seemingly unnecessary complication is purely for demonstration purpose. Perhaps the easiest way to solve the puzzle is to exploit more symmetry that is intrinsic to the puzzle. For example, one may bring tokens 2, 8, 5, and 11 to the right place in this order and get a simpler solution. It might as well be the case that making pairs of continuous tokens yields a simpler solution. These ideas are left as future work.