# THINK 37 (AND 37A) PROBLEM SET 3

**This set is due by noon on Thursday, October 18 in the manilla envelope in Otis Chodosh's mailbox.**

In this problem set and problem set 5 or 6, you will work out some distances known for thousands of years, and hopefully appreciate how remarkable it is that the ancients knew such things, and how we (and you) can know some things (in a more important way than simply being told by someone in a position of authority). In problem set 3, you will work out half of the problems and write them up. (The definition of half is as follows. There are 16 "problem parts" below, counting problem 5 as one, so do 8 of them. Feel free to do more, thereby saving yourself time on the later problem set.) In problem set 5 or 6, you will finish the job — rewriting the first half, and working out the other half. (The week's gap is so you have a chance for feedback.

*These problems were composed by Otis Chodosh based on a remarkable exposition by Nadir Jeevanjee.*

If you have never seen problems like this (most of you!), they may at first seem alarming. (Where do you even start?) You should brainstorm, and try things out. Pay attention (perhaps in future weeks) to how your mind is working, and how you have productive ideas. After you've had a chance to think about a problem, and think you have run out of steam, then talk with me or Otis. *Come to office hours! Also, Otis' email address is*

*ochodosh@math.stanford.edu.*

Then do it again and again. Talk to each other. Once you have figured out how it works, you are only halfway done; then you have to start writing down a proof.

**1. Eratosthenes Measures Earth's Radius.**

Eratosthenes knew that at noon in Syene, Egypt (which is roughly on the Tropic of Cancer) on June 21 (the summer solstice), the Sun was directly overhead. He repeated this measurement in Alexandria, and found that at the same time and date, the Sun was $7^\circ$ off from being directly overhead.

(a) Explain what factors affect the position of the Sun in the sky at "noon." Explain why noon is in quotation marks.

(b) One way that one might measure this deviation is by looking at the shadow of a (vertical) gnomon (stick). If Eratosthenes used a 1 meter stick, what shadow would the gnomon have cast?

*Date*: Friday, October 12, 2012.

(c) Given that Alexandria was 770 km north of Syene, explain how Eratosthenes arrived at a measurement of the Earth's radius, and compute the value.

(d) How might Eratosthenes have measured the distance between Syene and Alexandria?

(e) Comment on the accuracy of the measurement. For each measurement/observation that Eratosthenes made, discuss possible sources of error. Estimate how much this error might affect the final result.

(f) What other dates would have been convenient for Eratosthenes to perform his measurement, given that he had heard similar reports (what would he have heard?) about cities on the equator and the Tropic of Capricorn?

**2. Latitude, Earth's Tilt, and the Length of a Day.**

The angle between the tilt of Earth's axis and the Earth–Sun line varies throughout the year between $23.5^\circ$ and $-23.5^\circ$.

(a) How could you measure this tilt using only tools available to the ancients? *Hint*: Eratosthenes knew enough for this measurement, and he could have determined both his latitude and Earth's tilt without ever leaving Alexandria and without the knowledge that Syene was on the Tropic of Cancer or how far it was from Alexandria. *Follow-up question*: Why, then, did he need to know anything about Syene?

(b) Figures 1 and 2 are diagrams of how Earth's tilt and the latitude of an observer affect the length of a day. The plane marked $P$ is orthogonal to the Sun's rays and passes through the Earth's origin. Determine the length of a day in terms of $A$, the tilt of the Sun relative to Earth's axis of rotation and $B$, your latitude. *Hints*: How is this related to the angle $C$, labeled on Figure 2, given that a full day takes 24 hours? You should be able to solve for $C$ in terms of $A$ and $B$ with just basic trigonometry.

(c) Argue that the angle $A$ (the tilt of the Sun relative to Earth's axis) is well approximated by

$$A(t) = 23.5^\circ \times \cos\left(\frac{t}{365 \text{ days}} \cdot 360^\circ\right)$$

where $t$ is the number of days elapsed since the summer solstice (June 21). *Hints:* You may assume that you've already measured the $23.5^\circ$ maximum tilt. You may also assume that Earth's orbit around the Sun is a circle and not an ellipse.

(d) Combining the last two questions, how long was the Sun up today? Compare your answer to data from `http://aa.usno.navy.mil/data/docs/RS_OneDay.php` to determine its accuracy. Comment.

**3. Aristarchus Measures the Moon.**

(a) Aristarchus observed that the maximum length of a lunar eclipse is three hours. Using this and the fact that the Moon orbits Earth once a month to determine the distance from Earth to the Moon in terms of Earth's radius. Using Eratosthenes' result, determine

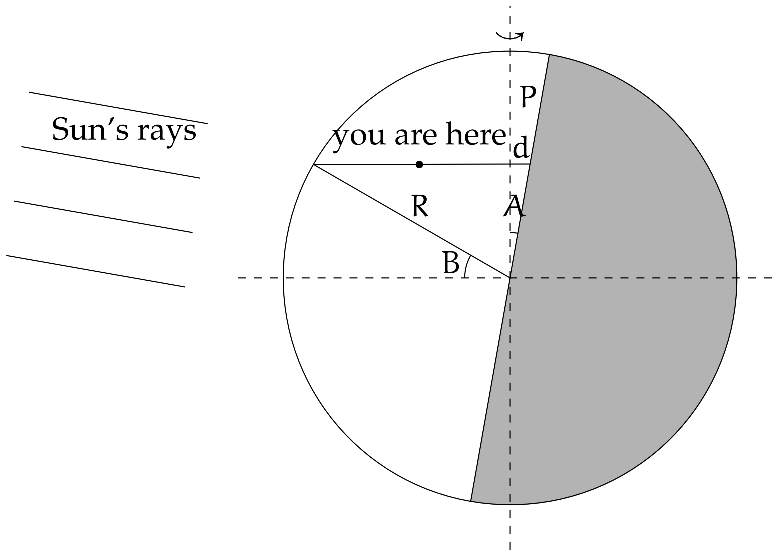


FIGURE 1. How Earth's tilt and an observer's latitude affect the length of a day.

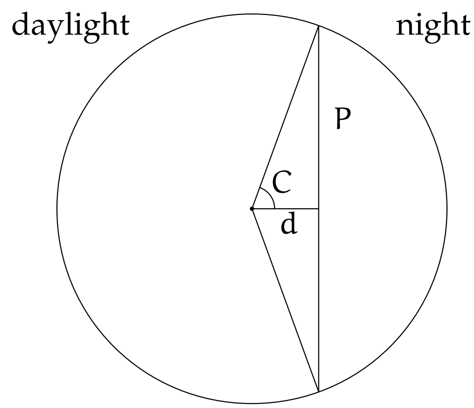


FIGURE 2. Overhead view of the circle of latitude, looking down at the north pole.

the distance to the Moon, and compare it to the currently accepted value. *Hint:* Why can we assume that Earth's shadow is approximately two Earth radii?

(b) He also observed that the Moon takes approximately two minutes to set. Why is this enough to determine the radius of the Moon, in terms of the distance to the Moon? Combine this with the above result to determine the radius of the Moon. Compare it to the current value.

**4. Aristarchus Measures the Sun.**

Aristarchus also observed that during a solar eclipse, the Moon almost perfectly covers the Sun.

(a) Given our knowledge about the Moon, what does this tell us about the ratio between the radius of the Sun and the distance to the Sun?

(b) Finally, Aristarchus needed to determine the distance from Earth to the Sun. One method for this is illustrated in Figure 3. Because of the finite distance from Earth to the Sun, the half Moon did not occur exactly halfway between a new Moon and a full Moon. Aristarchus measured that a half Moon occurred 12 hours before the midpoint of a new and full Moon. Using this value, what is the distance from Earth to the Sun? In fact, Aristarchus' measurements of the delay were way off (can you see why this would be a hard thing to measure?), and the currently accepted value is closer to 30 minutes. How does this affect the result? Compare your new result to the currently accepted distance.

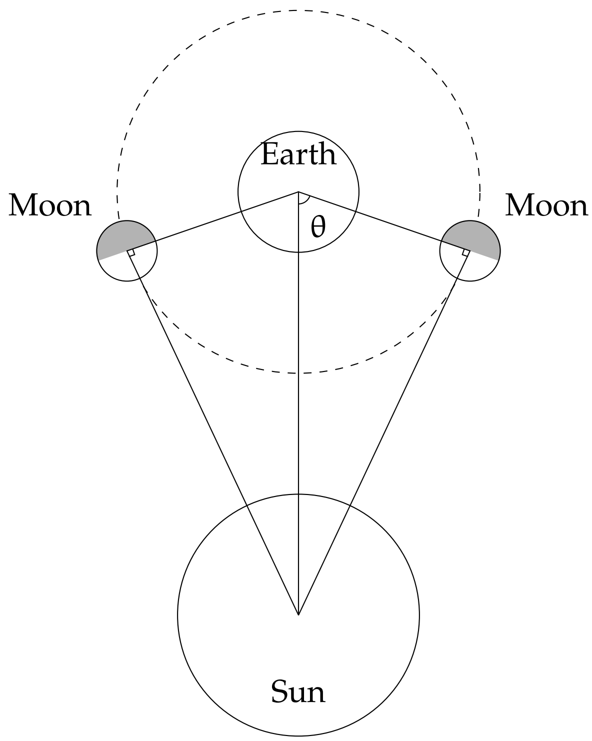


FIGURE 3. Measuring the distance from Earth to the Sun by the lag in half Moons.

(c) Using the results of the previous two parts, estimate the radius of the Sun. Compare with the accepted value.

**5. More open-ended problems.**

Do *one* of the following three, depending on what you feel up to.

(a) *(follow-up to problem 1)* Discuss the feasibility of your repeating Eratosthenes' measurement without leaving Stanford, and without any data about anything off campus. How might you to do this? How accurate would your measurements need to be? Is such accuracy achievable using modern technology?

(b) Discuss how the ancients (or modern astronomers) might have measured (currently measure) another distance/quantity not discussed above. Include your own calculations. Feel free to consult references, but be sure to appropriately cite your sources.

(c) *An Alternate Calculation by Aristarchus*

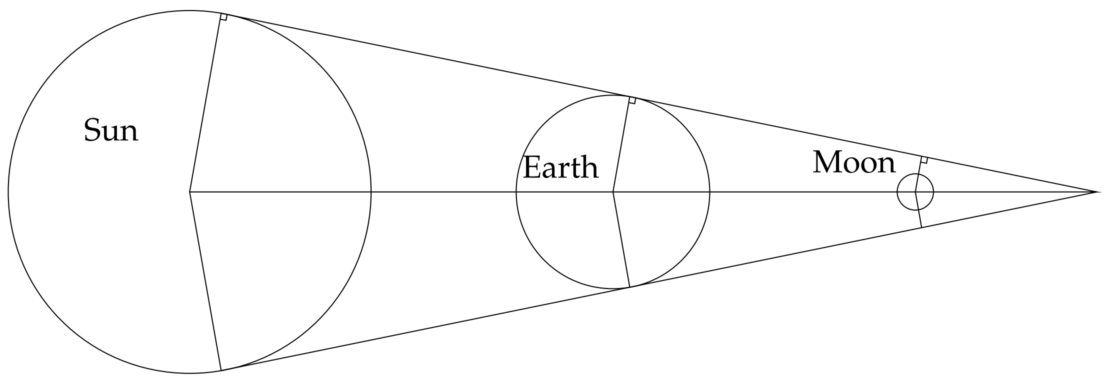


FIGURE 4. Outline of a diagram for the alternative method used by Aristarchus.

The following is an alternate method for calculating the Sun's radius. As above, by measuring the lag of the half Moon, Aristarchus could measure the ratio between the distance to the Moon and the distance to the Sun.

(c1) How is this related to the ratio of radii, given the observation that both bodies subtend roughly the same angle in the sky?

(c2) Now, Aristarchus' observation was that during a lunar eclipse, the time the Moon takes to enter the umbra (shadow) of the earth is roughly the same as the "duration of totality" (the time which the Moon is totally dark). Refer to Figure 4, and figure out how to use this information to determine an (approximate) relationship between the radius of the Moon and the radius of the shadow cast by Earth on the Moon. Then, using similar triangles, solve for the ratio between Earth's radius and the Sun's radius, in terms of the above information and the ratio between the Sun's radius and the Moon's radius. We've determined this in the last part, so you should now be able to compute the Sun's radius. Compare this to the answer calculated in the previous question, as well as with the currently accepted value. Comment on the differences in methods.

(c3) Aristarchus' least accurate result was the ratio between the distance from Earth to the Moon and the distance from Earth to the Sun. As discussed previously, this was because it was difficult to determine when the Moon was exactly half full. Can you think of a better way to measure this ratio? What if you have access to modern technology?