THINK 37 (AND 37A) PROBLEM SET 2

This set is due by noon on Thursday, October 11 in the manilla envelope in Otis Chodosh's mailbox.

Write solutions to four of the following eight problems. Pick problems suitable to your experience. If you are an experienced solver, concentrate on writing especially polished proofs.

If you have never seen problems like this (most of you!), they may at first seem alarming. (Where do you even start?) You should brainstorm, and try things out. Pay attention (perhaps in future weeks) to how your mind is working, and how you have productive ideas. After you've had a chance to think about a problem, and think you have run out of steam, then talk with me or Otis. *Come to office hours! Also, Otis' email address is*

ochodosh@math.stanford.edu.

Then do it again and again. Talk to each other. Once you have figured out how it works, you are only halfway done; then you have to start writing down a proof.

1. Suppose a, b, c, and d are rational numbers, and $a + b\sqrt{2} = c + d\sqrt{2}$. Show that $a = c$ and $b = d$.

2. (You can assume problem 1 when solving this.) Suppose $f(x)$ is a cubic polynomials with integer coefficients, say

$$f(x) = Ax^3 + Bx^2 + Cx + D.$$

Suppose a and b are rational numbers, and $x = a + b\sqrt{2}$ is a solution to $f(x) = 0$ (i.e. $f(a + b\sqrt{2}) = 0$). Show that $x = a - b\sqrt{2}$ is also a solution to $f(x) = 0$. (How do you think this can be generalized?)

3. The cubic $f(x) = x^3 - 17x^2 + 432x - 4324$ has three roots. What is their sum? (Hint: don't try to find them! I have told you that $f(x)$ factors as $(x - a)(x - b)(x - c)$ for some a, b, and c, and I have asked you for $a + b + c$.)

4. We have defined the real numbers as, essentially, decimals. An intelligent species on another planet has twelve fingers, and works in base 12. Explain why their "real numbers" are the "same" as ours. The challenge: explain it as convincingly as possible in a third of a page.

5. You and I play a game. You have five balls in a box, and each of them has a positive integer on it (perhaps 1, 10, 100, 1000, and 34). You will win if you can get rid of all the balls in your box. You can give me a ball, and then I can give back to you any number of

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balls I want (a non-negative integer), except that their numbers have to be smaller positive integers than the one you gave me. For example, if you give me ball 10, I can give you a million balls, all labeled 8 or 9. But if you give me ball 1, I can't give you anything back. Can you get rid of all the balls in the box (in a finite amount of time)?

6. Figure out how to define π purely "formally", without reference to the "real world". (Feel free to use things we have not discussed in class. I am most interested in the strategy.)

7. Suppose S is any set, and let P be the set of subsets of S. Show that there is no way of matching up the elements of S and P. In other words, show that function f from S to P, such that every subset of S is hit exactly by one element of f. For example, if $S = \{\text{dog}, \text{cat}\}$, then

$$P = \{\{\}, \{\text{dog}\}, \{\text{cat}\}, \{\text{dog}, \text{cat}\}\}.$$

Hint: if there is some function $f : S \to P$, ask yourself about the subset of S of elements s such that $s \notin f(s)$. (This becomes more interesting if S is infinite, but you should start thinking about the finite case. This will be trickier for Thursday's class, as we didn't discuss different sizes of infinities — but try it anyway...)

8. (Challenge problem, related to the ABC conjecture) The polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of $P(z) + 1$ and $Q(z) + 1$. Prove that $P(z)$ and $Q(z)$ are the same polynomial.