Motivic Integration

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In this talk I will speak about

1. Motivic measures
2. Motivic integration
3. Applications
4. Questions and comments.

There are excellent survey articles by A. Craw, E. Looijenga, Denef and Loeser: math.AG/9911179, 0066220, 0006050.

1 Arc space

Let X be a (possibly singular) algebraic variety of dimension d over $\mathbb{C}$. A morphism $\phi : \operatorname{Spec} \mathbb{C}[t]/(t^2) \to X$, which maps $\operatorname{Spec} \mathbb{C}$ to $x \in X$, determines a (Zariski) tangent vector at x.

A k-jet over x is a morphism $\phi : \operatorname{Spec} \mathbb{C}[t]/(t^{k+1}) \to X$ which maps $\operatorname{Spec} \mathbb{C}$ to x. Let $J_k(X)$ be the space of all k-jets and let $\pi_0^k : J_k(X) \to X$ be the map sending a k-jet over x to x.

For $n > k$, the quotient $\mathbb{C}[t]/(t^{n+1}) \to \mathbb{C}[t]/(t^{k+1})$ induces a morphism $\pi_k^n : J_n(X) \to J_k(X)$ and the projective limit

$$J_\infty(X) = \lim_{\to} J_k(X) = \{\gamma : \operatorname{Spec} \mathbb{C}[[t]] \to X\}$$

is called the arc space of X. We have the natural map $\pi_k : J_\infty(X) \to J_k(X)$. Locally, an arc $\gamma \in J_\infty(X)$ at a smooth point x is a d-tuple of power series in t.

The motivic integral is a "measure" $\mu(J_\infty(X))$ of the arc space. It is an invariant of X which retains useful information about X and its resolutions.

2 Motivic Measure

Let $\mathcal{V}_{\mathbb{C}}$ be the set of isomorphism classes of complex algebraic varieties. The Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ is the free abelian group generated by $\mathcal{V}_{\mathbb{C}}$ modulo $[V - V'] = [V] - [V']$ whenever V' is a closed subset of V. It becomes a ring with the multiplication $[V][V'] = [V \times V']$.

Let Y be a smooth variety of dimension d. Then $J_k(Y) \to Y$ is a $\mathbb{C}^{kd}$-bundle and hence $[J_k(Y)] = [Y][\mathbb{C}]^{kd}$. Let $\mathbb{L} = [\mathbb{C}]$ and $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ be the localization. Then

$$[J_k(Y)]\mathbb{L}^{-kd}$$

is independent of k. We define

$$\mu(J_\infty(Y)) = [J_k(Y)]\mathbb{L}^{-kd} = [Y] \in K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}].$$

More generally, when X is not smooth we make the following defintion.

Definition 2.1 *A subset A of $J_\infty(X)$ is* stable *if for some $k \in \mathbb{N}$ we have*

- *$\pi_k(A)$ is constructible and $A = \pi_k^{-1}\pi_k(A)$*
- *$\pi_{k+1}(A) \to \pi_k(A)$ is a (piecewise trivial) $\mathbb{C}^d$-fibration.*

The virtual dimension *of A is defined as $\dim \pi_k(A) - kd$. The measure of a stable set A is defined by*

$$\mu(A) = [\pi_k(A)]\mathbb{L}^{-kd}$$

which is independent of k.

When X is singular, $J_\infty(X)$ may not be stable. However, we can express $J_\infty(X)$ as a "limit" of stable sets. First decompose X as the disjoint union of smooth subvarieties by the vanishing order of the dth Fitting ideal J of Ω_X, i.e. if X is locally a subvariety of $\mathbb{C}^{d+l}$ given by $f_1 = 0, f_2 = 0, \cdots, f_l = 0$, then J is the ideal generated by the determinants of all $l \times l$ submatrices of $\left(\frac{\partial f_i}{\partial x_j}\right)$. This gives us an ideal sheaf $\mathcal{J}$ on X by restriction and hence a function

$$ord_{\mathcal{J}} : J_\infty(X) \to \mathbb{Z}$$

by $\gamma \to ord(\mathcal{J}|_\gamma)$. We have

$$J_\infty(X) = \bigsqcup_{m \in \mathbb{Z}} ord_{\mathcal{J}}^{-1}(m)$$

and $ord_{\mathcal{J}}^{-1}(m)$ is stable for each m. Also for $k \geq m$,

$$\dim \pi_k(ord_{\mathcal{J}}^{-1}(m)) - kd \to -\infty$$

as $m \to \infty$.

Let $M = K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ and $F_m M$ be the subspace generated by $[V]\mathbb{L}^{-k}$ such that $\dim V - k \leq m$. Define

$$\hat{M} = \widehat{\mathcal{V}_{\mathbb{C}}[\mathbb{L}^{-1}]} = \lim_{m} M/F_m M.$$

Definition 2.2 *A subset A of $J_\infty(X)$ is* measurable *if for any $m \in \mathbb{N}$ there exists a stable set A_m and a sequence $(C_i)_{0 \leq i \leq \infty}$ of stable sets such that*

$$A \triangle A_m \subset \cup_{i \in \mathbb{N}} C_i$$

and $\dim C_i < -m$, $\lim_{i \to \infty} C_i = -\infty$. *Define*

$$\mu(A) = \lim_{m \to \infty} \mu(A_m) \in \hat{M}.$$

$J_\infty(X)$ is measurable and the motivic integral of X is defined as

$$\mu(J_\infty(X)) \in \hat{M}.$$

3 Motivic integration

Let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf. It defines a function

$$ord_{\mathcal{I}} : J_\infty(X) \to \mathbb{N} \cup \{\infty\}$$

by $\gamma \to ord(\mathcal{I}|_\gamma)$. For each $m \in \mathbb{N} \cup \infty$, $ord_{\mathcal{I}}^{-1}(m)$ is a measurable set and $ord_{\mathcal{I}}^{-1}(\infty)$ is measure zero. If $Z \subset X$ is a closed subvariety we also denote $ord_{\mathcal{I}_Z}$ by ord_Z.

Define the motivic integral of $(X, \mathcal{I})$ by

$$\int_{J_\infty(X)} \mathbb{L}^{-ord_{\mathcal{I}}} = \sum_{m \in \mathbb{Z}} \mu(ord_{\mathcal{I}}^{-1}(m))\mathbb{L}^{-m}.$$

Theorem 3.1 *Let $\rho : Y \to X$ be a surjective birational morphism. Let $\mathcal{J}$ be the Jacobian ideal of ρ, i.e. the 0th Fitting ideal of $\Omega_{Y/X}$ (i.e. $K_Y - \rho^* K_X$). If A is a measurable subset of $J_\infty(Y)$ such that $\rho_\infty|_A$ is injective, then $\rho_\infty(A)$ is measurable and*

$$\mu(\rho_\infty(A)) = \int_A \mathbb{L}^{-ord_{\mathcal{J}}} d\mu.$$

Proof Careful stratification of A and $\rho_\infty(A)$.

Corollary 3.2 *Let $\rho_i : Y_i \to X$ be two resolutions of X and let $W_i = K_{Y_i} - \rho_i^* K_X$ for $i = 1, 2$. Then*

$$\int_{J_\infty(Y_1)} \mathbb{L}^{-ord_{W_1}} = \int_{J_\infty(Y_2)} \mathbb{L}^{-ord_{W_2}}.$$

Suppose $\rho : Y \to X$ be a resolution such that $K_Y = \rho^* K_X + \sum_{i=1}^n a_i D_i$ where $a_i > -1$ and D_i are normal crossing divisors. For $J \subset \{1, 2, \cdots, n\}$, let $D_J = \cap_{j \in J} D_j$ and $D_J^0 = D_J - \cup_{j \notin J} D_j$.

Theorem 3.3

$$\mu(J_\infty(X)) = \int_{J_\infty(Y)} \mathbb{L}^{-ord_{\mathcal{J}}} = \sum_J [D_J^0] \prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j + 1} - 1}.$$

Proof Careful stratification of $J_\infty(Y)$ with respect to the values of $ord_{\mathcal{J}}$.

Corollary 3.4 *If $\rho : Y \to X$ is a crepant resolution, i.e. $K_Y = \rho^* K_X$, then $\mu(J_\infty(X)) = \mu(J_\infty(Y)) = [Y]$.*

4 Stringy E-fuction

Let X be a complex algebraic variety. Then the E-polynomial is

$$E(X) = \sum (-1)^k h^{p,q}(H_c^k(X;\mathbb{C}))u^p v^q$$

which defines a map $E : M = K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}] \to \mathbb{Z}[u, v, (uv)^{-1}]$. This gives rise to the stringy E-function

$$E_{st}(X; u, v) = E(\mu(J_\infty(X))) = \sum_J E(D_J^0) \prod_{j\in J} \frac{uv - 1}{(uv)^{a_j+1} - 1}.$$

The first major application is the proof of Kontsevich's theorem.

Theorem 4.1 *(Kontsevich's theorem) Let $\rho_i : Y_i \to X$ for $i = 1, 2$ be two crepant resolutions. Then $E(Y_1) = E(Y_2)$, i.e. $h^{p,q}(Y_1) = h^{p,q}(Y_2)$.*

Remark 4.2 Mirror symmetry test: If V and V^* are mirror varieties, then $E_{st}(V; u, v) = u^d E_{st}(V^*; u^{-1}, v)$. This is proved for Batyrev mirrors coming from dual polytopes.

The second application is to McKay correspondence. For orbifolds, the stringy E-function is not the same as the orbifold E-function. Consider for example $\mathbb{P}^2 = (\mathbb{P}^1)^2/\mathbb{Z}_2$. We have

$$E_{st}(\mathbb{P}^2) = 1 + uv + (uv)^2$$

$$E_{orb}((\mathbb{P}^1)^2, \mathbb{Z}_2) = 1 + uv + (uv)^2 + (uv)^{\frac{1}{2}}(1 + uv).$$

The discrepancy comes from the codimension 1 branches.

Let G be a finite group which acts regularly on a smooth algebraic variety V. Let $X = V/G$ and $\Delta_1, \cdots, \Delta_k \subset X$ be the codimension 1 irreducible ramification locus. Let ν_i be the order of the cyclic group corresponding to Δ_i. Define

$$\Delta_X = \sum_{i=1}^k \frac{\nu_i - 1}{\nu_i} \Delta_i.$$

If $\rho : Y \to X$ is a resolution such that $K_Y = \rho^*(K_X + \Delta_X) + \sum_{i=1}^k a_i D_i$, then we define the stringy E-function of the pair

$$E_{st}(X, \Delta_X) = E\Big(\int_{J_\infty(X)} \mathbb{L}^{ord_{\Delta_X}}\Big) = \sum_J E(D_J^0) \prod_{j\in J} \frac{uv - 1}{(uv)^{a_j+1} - 1}.$$

Theorem 4.3 *(Batyrev)*

$$E_{st}(X, \Delta_X) = E_{orb}(V, G).$$

Corollary 4.4 $e_{st}(X, \Delta_X) = e_{orb}(V, G)$

If $V = \mathbb{C}^n$ and $G \subset SL(n, \mathbb{C})$ is a finite subgroup, then $\Delta_X = 0$ and thus

$$e_{st}(X, \Delta_X) = e_{st}(X) = e(Y) \text{ for any crepant resolution } Y \to X.$$

Also, $e_{orb}(\mathbb{C}^n, G)$ is equal to the number of conjugacy classes in G. Hence, $e(Y)$ is the number of conjugacy classes in G for any crepant resolution $Y \to X$.

5 Examples

1. $X = \mathbb{P}^n /\!\!/ \mathbb{C}^*$ where $\mathbb{C}^*$ acts on $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^p \times \mathbb{C}^q \times \mathbb{C}^r)$ with weight $1, -1, 0$, $p < q$, respectively on each component.

$$E_{st}(X) = \frac{1 - (uv)^p - (uv)^{p+r} + (uv)^{n+1}}{(1 - uv)^2} = IE(X).$$

2. $X = (\mathbb{P}^1)^{2n} /\!\!/ SL(2)$.

$$E_{st}(X) = \frac{(1+uv)^{2n}}{1-(uv)^2} - \sum_{n<r\leq 2n} \binom{2n}{r} \frac{(uv)^{r-1}}{1-uv} - \frac{1}{2}\binom{2n}{n}\frac{(uv)^{n-1}}{1-uv} = IE(X)$$

3. $X \subset \mathbb{C}^4$ hypersurface $x^2 + y^2 + z^2 + t^3 = 0$.

$$E_{st}(X) = (uv)^2 \frac{(uv)^3 + (uv)^2 + 2uv + 1}{(uv)^2 + uv + 1}$$

4. Toric varieties

5. Moduli space $\mathcal{M}(2, 0)$

6 Problems and comments

1. What is the relation with the intersection cohomology? When do we get $E_{st}(X) = IE(X)$? Of course, when X admits a small resolution, it is true. But the examples above indicate it is true for a larger class of varieties.

2. Compute the motivic integral of the moduli space $\mathcal{M}(r, d)$ of vector bundles of degree d rank r over a Riemann surface of genus g when d and r are not coprime. Done for $d = 0, r = 2, g = 3$.

3. How good is the motivic integral as an invariant?

4. Mustata used motivic integral to show that when X is a local complete intersection, X has rational singularities iff $J_m(X)$ is irreducible for all m. (Embed X into a smooth variety Y. Resolve the singularities and compute the motivic integral of the pair (Y, X).)