POLYA PROBLEM-SOLVING SEMINAR WEEK 6: PROBABILITY AND EXPECTED VALUES

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The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

The Problems.

Sample 1. If $0 < p < 1$, show that $\sum p^n = 1/(1-p)$ by thinking of p as a probability.

Sample 2. You have a coin that turns up heads with probability p. How many flips do you expect to make before you see your first head? Use this to simplify $\sum nx^n$.

Sample 3. Prove the identity

$$1 + \frac{n}{m+n-1} + \cdots + \frac{n(n-1)\cdots 1}{(m+n-1)(m+n-2)\cdots m} = \frac{m+n}{m}.$$

Sample 4. A stick is broken in two random places. What are the odds that the three pieces can form the sides of a triangle?

Sample 5. Show that the shadow of a unit line segment in the plane in a random direction is $2/\pi$. (What happens in higher dimensions?)

1. *(trick question)* A bag contains 2007 red balls and 2007 black balls. We remove two balls at a time repeatedly and

(i) discard them if they are of the same color,
(ii) discard the black ball and return to the bag the red ball if they are of different colors.

What is the probability that this process will terminate with one red ball in the bag? (Gelca and Andreescu, problem 910)

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2. By considering n flips of a fair coin, show:

a)

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

b)

$$1\binom{n}{1} + 2\binom{n}{2} + \ldots + n\binom{n}{n} = n2^{n-1}$$

c)

$$1\binom{n}{0} + x\binom{n}{1} + x^2\binom{n}{2} + \ldots + x^n\binom{n}{n} = (x+1)^n$$

3. What is the probability that three points selected at random on a circle lie on a semicircle? (Gelca and Andreescu, problem 931)

4. Two evenly matched teams play in the world series, a best of seven competition in which the competition stops as soon as one team has won four games. Is the world series more likely to end in six or seven games?

5. Let π be a random permutation of the numbers $1, 2, \ldots, n$. What is the expected length of the cycle containing 1?

6. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots? (2002B1)

7. Let $p_n(k)$ be the number of permutations of $\{1, 2, \ldots, n\}$ which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k p_n(k) = n!$$

(from 1987)

8. You toss n coins, and you win if you turn up an even number of heads. Otherwise, Bob Hough takes your lunch money.

(a) Show that your odds of winning are 50% if all the coins are fair coins.
(b) Better yet, show that your odds of winning are 50% if at least *one* of the coins is fair.
(c) In fact this is best possible: show that your odds of winning are *not* 50% if *none* of the coins are fair. (Your odds of winning are then in fact better than 50% if and only if an even number of coins are weighted toward heads.)

9. Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers. (1993B3)

10. Prove that the sum

$$\sum_{n=k}^{\infty} \binom{n}{k} 2^{-n}$$

is independent of k and determine its value.

11. For a list of numbers $a_1, a_2, \ldots, a_h$ let $d(a_1, \ldots, a_h)$ be the number of distinct values in the list. For example, $d(1,3,2,1) = 3$. Show that

$$\frac{1}{n^h} \sum_{a_1, a_2, \ldots, a_h = 1}^{n} \left(1 - \frac{d(a_1, \ldots, a_h)}{n}\right) = \left(\frac{n-1}{n}\right)^h .$$

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