POLYA PROBLEM-SOLVING SEMINAR WEEK 5: INVARIANTS AND MONOVARIANTS

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The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

The Problems.

(These problems are from many sources. Several are from Gabriel Carroll's talk at the Berkeley Math Circle.)

Sample 1. Two players take turns breaking up an $m \times n$ chocolate bar. On a given turn, a player picks a rectangular piece of chocolate and breaks it into pieces along the subdivisions between its squares. The player who makes the last break wins. Does one of the players have a winning strategy?

Sample 2. Given n red points and n blue points in the plane, show that we can draw n non-intersecting line segments, each having one red endpoint and one blue endpoint.

Sample 3. Some people are in a building with several rooms. Each minute a person leaves a room and moves to another that has at least as many people. Show that eventually all of the people are in a single room.

Sample 4. The first problem-of-the-week: Find positive integers n and $a_1, \ldots, a_n$ such that $a_1 + \ldots + a_n = 100$ and the product is as large as possible.

1. n soccer teams participate in a knock-out tournament. At each stage, the remaining teams pair up and play games, and the losers are eliminated. (If there are an odd number of teams at a stage, one team is chosen by lot to get "bye" and not play.) At the end, the Stanford team is declared the winner. How many total games have been played??

2. Show that the monovariant $\sum 1/(n_i + 1)$ also works in Sample 3.

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3. N men and N women are distributed among the rooms of a mansion. They move among the rooms according to the rules: either

- a man moves from a room with more men than women (counted before he moves) into a room with more women than men, or
- a woman moves from a room with more women than men into a room with more men than women.

Show that eventually people will stop moving.

4. A deck contains cards numbered $1, 2, ..., n$ in random order. Repeat: when the card on the top of the deck is numbered k, reverse the order of the first k cards. Show that 1 is eventually on the top of the deck.

5. Suppose you have two sets of n real numbers, $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$. Show that if σ is any permutation of the numbers 1 through n, then

$$x_1y_n+x_2y_{n-1}+\cdots+x_{n-1}y_2+x_ny_1 \le x_1y_{\sigma(1)}+x_2y_{\sigma(2)}+\cdots+x_ny_{\sigma(n)} \le x_1y_1+x_2y_2+\cdots+x_ny_n.$$

6. Some people are in a building with several rooms. Each minute, as long as the people are not all in the same room, the following happens: one or more people, all of whom were previously in the same room as each other, leave that room. At least one of them goes to a new room with more people than the original room; the rest may go anywhere. Show that eventually all of the people are in a single room.

7. Some number of frogs are squatting on a row of 2000 lily pads in a swamp. Each minute, if there are two frogs on the same lily pad, and this pad is not at either end of the row, the two frogs may jump to two adjacent lily pads (in opposite directions). Prove that this process cannot be repeated forever.

8. This problem appeared on the 1986 IMO: "To each vertex of a regular pentagon an integer is assigned, such that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x+y, -y, z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps." Joe Keane received a special prize for coming up with the monovariant

$$\begin{gathered}(|a|+|b|+|c|+|d|+|e|)\\ +(|a+b|+|b+c|+\cdots+|e+a|)\\ +(|a+b+c|+|b+c+d|+\cdots+|e+a+b|)\\ +(|a+b+c+d|+\cdots+|e+a+b+c|)\end{gathered}$$

Figure out his solution! (Then: Can you get an inkling of where it came from?)

9. *The problem of the week.* You play a game against Bob Hough. You have a very big urn, and at the start of the game, there are twenty balls, two each numbered one through ten. At each turn, you throw out one ball, and Bob dumps in any (finite) number of his choice of balls with smaller (positive integer) labels. For example, if you throw out a ball labeled ten, Bob could dump in a million balls labeled four. But if you throw out a ball labeled one, then Bob can't do anything. You win if you can empty the urn. Can you win? What is your strategy?

10. The country of Philatelia is founded for the pure benefit of stamp-lovers. Each year the country introduces a new stamp, for a denomination (in cents) that cannot be achieved by any combination of older stamps. Show that at some point the country will be forced to introduce a 1-cent stamp, and the fun will have to end.

11. On an $n \times n$ board there are n^2 squares, $n-1$ of which are infected. Each second, any square that is adjacent to at least two infected squares becomes infected. Show that at least one square always remains uninfected.

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