POLYA PROBLEM-SOLVING SEMINAR WEEK 4: COMPLEX NUMBERS

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The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on. If you would like to practice with the Pigeonhole Principle or Induction (a good idea if you haven't seen these ideas before), try those problems.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

The Problems.

Sample 1. Show that if a, b, n are positive integers then there exist integers x, y such that $(a^2 + b^2)^n = x^2 + y^2$.

Sample 2. Suppose $ABCD$ is a convex plane quadrilateral, and squares are erected on the outside of each edge, with centers $EFGH$. Show that EG is perpendicular to FH, and both segments have the same length.

Sample 3. Show that $\cos 36^\circ = \tau/2$.

1. (a) Calculate $\sum_{n=1}^{360} \sin n^\circ$. (b) Calculate $\sum_{n=1}^{90} \sin n^\circ$. (c) Calculate $\sum_{n=1}^{360} \sin(x + n^\circ)$ for each x.

2. Suppose ABC is a plane triangle, and equilateral triangles ABE, BCF, and CAG are erected on the outside of the triangle. Show that the centers of these three triangles themselves form an equilateral triangle.

3. Suppose $f(x)$ is a polynomial with real co-efficients such that $f(x) \geq 0$ for all x. Show that there exist polynomials $g(x)$ and $h(x)$ with real coefficients such that $f(x) = g(x)^2 + h(x)^2$.

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4. Curves A, B, C, and D are defined in the plane as follows:

$$\begin{aligned} A &= \left\{ (x,y) : x^2 - y^2 = \frac{x}{x^2+y^2} \right\}, \\ B &= \left\{ (x,y) : 2xy + \frac{y}{x^2+y^2} = 3 \right\}, \\ C &= \{ (x,y) : x^3 - 3xy^2 + 3y = 1 \}, \\ D &= \{ (x,y) : 3x^2y - 3x - y^3 = 0 \}. \end{aligned}$$

(The equations defining A and B are indeterminate at $(0,0)$. The point $(0,0)$ belongs to neither.) Prove that $A \cap B = C \cap D$. (1987A1)

5. Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx$. For which integers m, $1 \le m \le 10$, is $I_m \neq 0$? (1985A5)

6. There is a 12-sided polygon inscribed in a unit circle (radius = 1). If you multiply the lengths of all sides and all diagonals of this polygon what will be the result? (proposed by Meng-Hsuan Wu, from the PuzzleUp competition in 2006)

7. Given a point P_0 in the plane of the triangle $A_1A_2A_3$. Define $A_s = A_{s-3}$ for all $s \ge 4$. Construct a set of points P_1, P_2, P_3, ...such that P_{k+1} is the image of P_k under a rotation center A_{k+1} through an angle 120° clockwise for $k = 0, 1, 2, \ldots$. Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral. (IMO1986#5)

8. Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.) (1989A3)

9. For an integer $n \ge 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k. (1999B5)

10. Suppose f and g are nonconstant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y,

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x. (1991B2)

(Bob Hough will present two more complex problems in the masterclass.)

Problem of the Week: the "Lights-out" game. Suppose $n \ge 2$ light bulbs are arranged in a row, numbered 1 through n. Under each bulb is a button. Pressing the button will change the state of the bulb above it (from on to off or vice versa), and will also change the state's neighbors. (Most bulbs have two neighbors, but the bulbs on the end have only one.) The bulbs start off randomly (some on and some off). For which n is it guaranteed to be possible that by flipping some switches, you can turn all the bulbs off?