POLYA PROBLEM-SOLVING SEMINAR WEEK 3: RECURRENCES

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The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on. If you would like to practice with the Pigeonhole Principle or Induction (a good idea if you haven't seen these ideas before), try those problems.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Sample recurrence write-up.

Problem. Solve the linearly recurrent equation $f_n = 5f_{n-1} - 6f_{n-2}$, $f_0 = 0$, $f_1 = 1$.

Solution. The characteristic equation for this recurrence is $t^2 - 5t + 6 = 0$, which has solutions $t = 2$ and $t = 3$ (each with multiplicity 1. Thus the solutions to the recurrence are all of the form $A2^n + B3^n$ (for constants A and B). Using the values at $n = 0$ and $n = 1$, we find that $A = -1$ and $B = 1$. Hence the solution is $f_n = 3^n - 2^n$.

The Problems.

1. Here are a bunch of problems that show recurrences from different points of view.

(a) The sequence $q_1, q_2, \ldots$ satisfies $q_n = 3q_{n-2} - 2q_{n-3}$, and $q_0 = 0$, $q_1 = 3$, $q_2 = 11$. Find a general formula for q_n.

(b) What is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$?

(c) The sequence $r_1, r_2, \ldots$ satisfies $r_n = (5/2)r_{n-1} - r_{n-2}$, and $r_1 = 2003$. Suppose the sequence converges to a finite real number. Find r_2.

(d) The sequence $G_0, G_1, G_2, \ldots$ consists of every other Fibonacci number. Show that there is a linear recursion (e.g. of the form $G_n = aG_{n-1} + bG_{n-2}$). (Follow-up: How about a sequence consisting of every *tenth* Fibonacci number. How do you know there's a recursion? Harder: With integer coefficients?)

Date: Monday, October 22, 2007.

(e) Use the theory of linear recursive sequences to find a formula for the sequence $s_0 = 1$, $s_1 = 2$, $s_n = s_{n-2}$. What do you observe? Now try a sequence with period four, such as $t_0 = 1$, $t_1 = 0$, $t_2 = 0$, $t_3 = 0$.

(f) Find a recurrence satisfied by $f_n = 3^n + 4^{n+1}$.

(g) Find a recurrence satisfied by all cubic polynomials.

(h) Suppose $f_2 = 2$, and $f_n = -2f_{n-1} - 4f_{n-2}$. Find f_{2003}. (It looks like there isn't enough information to solve this problem.)

(i) Find a length two recurrence satisfied by $C_n = \cos n°$.

(j) *(Ordinary differential equations with constant coefficients — intended only for those who haven't seen these before!)* Solve the differential equation $f''(x) = 5f'(x) - 6f(x)$, with initial conditions $f(0) = 0$, $f'(0) = 1$. (Translation: develop the general theory of such equations.) *Hint:* use the characteristic equation to guess two solutions. To guess a solution, examine a simpler equation of the same type, such as $g'(x) = 7g(x)$, and find its solutions. (What would happen if the differential equation had a repeated root? Try it with $f''(x) = 2f'(x) - f(x)$.)

2. Suppose f_n is a sequence of rational numbers, with f_0 and f_1 not both zero, such that $f_n = f_{n-1} + f_{n-2}$. Show that f_n is unbounded as $n \to \infty$. Is this still true if the condition of rationality is removed?

3. A gambling graduate student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring exactly n points at some time in a sequence of n tosses is $(2 + (-1/2)^n)/3$. (Hint: Let P_n denote the probability of scoring exactly n points at some time. Express P_n in terms of P_{n-1}, or in terms of P_{n-1} and P_{n-2}. Use this linear recursion to give an inductive proof. Even better hint, useful in many circumstances: you've been given the answer, so reverse-engineer the recursion, and then try to prove it.)

4. Solve the double recurrence

$$\begin{aligned} f_n &= f_{n-1} - 3g_{n-1} \\ g_n &= -3f_{n-1} + 9g_{n-1} \end{aligned}$$

(One possible approach: find a recurrence involving just f or g.) If you solve this and don't use eigenvalues and eigenvectors (you don't need them!), please tell me, and I'll teach you about them.

5. Show that $m|n$ if and only if $F_m|F_n$. (This is useful. For example: (i) Show that if n divides a single Fibonacci number that it will divide infinitely many Fibonacci numbers. (ii) Let $0 < i_1 < i_2 < ... < i_n$ be n integers. Prove that there exists a Fibonacci number F_N so that the least common multiple, $[F_{i_1}, F_{i_2}, ..., F_{i_n}]$ divides F_N.)

6. (This isn't a *linear* recurrence question, but it *is* a neat recurrence question.)

(a) Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$. Find a recurrence relation for I_n.

(b) Show that

$$I_{2n} = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times (2n)} \cdot \frac{\pi}{2}.$$

(c) Show that

$$I_{2n+1} = \frac{2 \times 4 \times 6 \times \cdots \times (2n-2)}{1 \times 3 \times 5 \times \cdots \times (2n-1)}.$$

(Fun follow-up: Write these formulas in terms of factorials. Hint: Can you see why $1 \times 3 \times \cdots \times (2n-1) = (2n)!/(2^n n!)$? Then try plugging $n = 1/2$ into the formula you get for (b); what do you get for $(1/2)!$? What's that $\sqrt{\pi}$ doing there?!)

7. Define a *selfish* set to be a set that has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, ..., n\}$ that are minimal selfish sets, that is, selfish sets none of whose proper subsets are selfish. (Putnam 1996B1)

8. For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$. (1997A6)

(On a related note: a useful fancy theorem is the *Perron-Frobenius Theorem*. In its simplest form, this theorem states that any matrix with positive entries has a unique eigenvector with positive entries, and that the corresponding eigenvalue has multiplicity one and has absolute value strictly greater than that of any other eigenvalue. Here is a fun application, which I heard from Mira Bernstein, director of the Canada-USA Mathcamp, when she was a post-doc here: The Seven Dwarfs are sitting around the breakfast table; Snow White has just poured them some milk. Before they drink, they perform a little ritual. First, Dwarf #1 distributes all the milk in his mug equally among his brothers' mugs (leaving none for himself). Then Dwarf #2 does the same, then Dwarf #3, #4, etc., finishing with Dwarf # 7. At the end of the process, the amount of milk in each dwarf's mug is the same as at the beginning! If the total amount of milk is 42 ounces, how much milk did each of them originally have?)

9. Suppose there are $2n$ people in a circle; the first n are "good guys" and the last n are "bad guys". Show that there is always an integer m (depending on n) such that, if we go around the circle executing every mth person, all the bad guys are first to go. (For example, when $n = 3$ we can take $m = 5$; when $n = 4$ we can take $m = 30$.) (p.frm-e0 of Graham-Knuth-Patashnik's *Concrete Mathematics*)

10. Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}.$$

(Putnam 2006B6)

This handout can be found at **http://math.stanford.edu/~vakil/putnam07/**

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