POLYA PROBLEM-SOLVING SEMINAR WEEK 1: INDUCTION AND PIGEONHOLE

RAVI VAKIL

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on. If you would like to practice with the Pigeonhole Principle or Induction (a good idea if you haven't seen these ideas before), try those problems.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Mathematical induction. Let a be an integer, and $P(n)$ a proposition (statement) about n for each integer $n \geq a$. The principle of mathematical induction states that if

(i) $P(a)$ is true, and
(ii) for each integer $k \geq a$, $P(k)$ true implies $P(k+1)$ true, then $P(n)$ is true for all integers $n \geq a$.

This principle enables us, in two simple steps, to prove an *infinite* number of propositions. It works best when you have observed a pattern and want to prove it.

The pigeonhole principle. If $kn+1$ objects ($k \geq 1$) are distributed among n boxes, one of the boxes will contain at least $k+1$ objects.

W1. Show that
$$1 + 2 + 3 + \cdots + n = n(n+1)/2.$$

W2. Show that there are two people in New York City, who are not totally bald, who have the exact same number of hairs on their head.

W3. Consider any five points $P_1, \dots, P_5$ in the interior of a square S of side length 1. Show that one can find two of the points at distance at most $1/\sqrt{2}$ apart. Show that this is the best possible.

Date: Monday, October 8, 2007.

W4. Prove that in any group of six people there are either three mutual friends or three mutual strangers. (Possible follow-up: Find some n so that in any group of n people there are either four mutual friends or four mutual strangers.)

1. Find a formula for the sum of the first n odd numbers.

2. Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet in a point (e.g. $f(4) = 11$). Find a formula for $f(n)$.

3. (Zeitz p. 51) Let $\mathbb{N}$ be the set of positive integers. Define f on n by $f(1) = 1$, $f(2n) = f(n)$ and $f(2n+1) = f(n) + 1$. Prove that $f(n)$ is the number of 1's in the binary representation of n.

4. Show that the decimal representation of a rational number must eventually repeat. (Zeitz p.91)

5 Let $P(x)$ be a polynomial with integer coefficients and degree at most n. Suppose that $|P(x)| < n$ for all $|x| < n^2$. Show that P is constant.

6. Prove that any 55 element subset of the integers in $[1, 100]$ contains elements that differ by 10, 12 and 13 but need not contain elements differing by 11. (Larson p. 98)

7. Let F_i denote the ith term in the Fibonacci sequence. Prove that $F_{n+1}^2 + F_n^2 = F_{2n+1}$. (Larson p. 72)

8. Prove the *arithmetic mean - geometric mean inequality* (AM-GM): Suppose $a_1, \dots, a_n$ are n positive real numbers. Then

$$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 \cdots a_n)^{1/n}.$$

(Call this statement AMGM(n).) Prove AMGM(n) for all n as follows.

(a) Prove it for $n = 1$ and $n = 2$.
(b) If it is true for $n = k$, prove that it is true for $n = k - 1$. (*Hint:* substitute $a_n = (a_1 + \cdots + a_{n-1})/(n-1)$ in AMGM(n) and see what happens.)
(c) If it is true for $n = k$, prove that it is true for $n = 2k$.
(d) Conclude!

9. Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume, for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$. (2000B1)

10. Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries. (1994A4)

This handout can be found at **http://math.stanford.edu/~vakil/putnam07/**

E-mail address: vakil@math.stanford.edu