PROBLEM-SOLVING MASTERCLASS WEEK 7

1. *(Wythoff's Game)* Two players play a game that uses two piles of counters. On a player's turn, that player may remove any number of counters (1 or more) from one of the piles, or remove a single number of counters (1 or more) from both piles. If a player has no legal move, he loses. Show that the second player to move wins if and only if the numbers of counters in the first pile and the second pile are $\lfloor n\phi \rfloor$ and $\lfloor n\phi^2 \rfloor$, or vice versa, where n is a non-negative integer and $\phi = \frac{1+\sqrt{5}}{2}$. (Nathaniel Shar)

2. Find all $e \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ and a sequence $a_0, a_1, a_2, \dots, a_n$ such that

(i) $a_i \in \mathbb{N}$ for all i,
(ii) $a_0 = 0$,
(iii) $2a_i \le a_{i+1} \le 2a_i + e$ for all i,
(iv) $\sum a_i = N$,

where i is all integers that make the expressions make sense. (Evan O'Dorney)

3. For an integer $n \ge 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k. (Kiat Chuan Tan, 1999B5)

4. For a positive real number α, define

$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \dots\}.$$

Prove that $\{1, 2, 3, \dots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha), S(\beta)$ and $S(\gamma)$. (Olena Bormashenko, 1995B6)

5. There is a 12-sided polygon inscribed in a unit circle (radius = 1). If you multiply the lengths of all sides and all diagonals of this polygon what will be the result? (Meng-Hsuan Wu, from the PuzzleUp competition in 2006)

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