1. Let \( p(z) \) be a polynomial of degree \( n \), all of whose zeros have absolute value 1 in the complex plane. Put \( g(z) = p(z)/z^{n/2} \). Show that all zeros of \( g'(z) = 0 \) have absolute value 1. (Bob Hough, 2005A3)

2. A Dyck \( n \)-path is a lattice path of \( n \) upsteps \((1, 1)\) and \( n \) downsteps \((1, -1)\) that starts at the origin \( O \) and never dips below the \( x \)-axis. A return is a maximal sequence of contiguous downsteps that terminates on the \( x \)-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.

![Dyck Path Diagram]

Show that there is a one-to-one correspondence between the Dyck \( n \)-paths with no return of even length and the Dyck \((n - 1)\)-paths. (Olena Bormashenko, 2003A5)

3. Let \( n \) be a positive odd integer and let \( \theta \) be a real number such that \( \theta/\pi \) is irrational. Set \( a_k = \tan(\theta + k\pi/n) \), \( k = 1, 2, \ldots, n \). Prove that

\[
\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}
\]

is an integer, and determine its value. (Bob Hough, 2006A5)

4. An \( m \times n \) checkerboard is colored randomly: each square is independently assigned red or black with probability \( 1/2 \). We say that two squares, \( p \) and \( q \), are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at \( p \) and ending at \( q \), in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than \( mn/8 \). (Jackson Gorham, 2004A5)

5. Show that for any positive integer \( n \), there is an integer \( N \) such that the product

\[ x_1 x_2 \cdots x_n \]

can be expressed identically in the form

\[
 x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n
\]

where the \( c_i \) are rational numbers and each \( a_{ij} \) is one of the numbers \(-1, 0, 1\). (Ryan Williams, 2004A4)

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