PUTNAM PROBLEM-SOLVING SEMINAR WEEK 7: GENERAL PROBLEM-SOLVING

RAVI VAKIL

The Rules. These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

1989A1. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?

1998A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

1987A2. The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the $10^{n\text{th}}$ digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100^{th} digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

1986A3. Evaluate $\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2 + n + 1)$, where $\operatorname{Arccot} t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot \theta = t$.

1995A4. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k - 1 \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

1989A5. Let m be a positive integer and let $\mathcal{G}$ be a regular $(2m+1)$-gon inscribed in the unit circle. Show that there is a positive constant A, independent of m, with the following

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property. For any point p inside $\mathcal{G}$ there are two distinct vertices v_1 and v_2 of $\mathcal{G}$ such that

$$\big|\,|p - v_1| - |p - v_2|\,\big| < \frac{1}{m} - \frac{A}{m^3}.$$

Here $|s - t|$ denotes the distance between the points s and t.

1986A6. Let $a_1, a_2, \ldots, a_n$ be real numbers, and let $b_1, b_2, \ldots, b_n$ be distinct positive integers. Suppose there is a polynomial $f(x)$ satisfying the identity

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \ldots, b_n$ and n (but independent of $a_1, a_2, \ldots, a_n$).

1995B1. For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

1996B1. Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

1994B2. For which real numbers c is there a straight line that intersects the curve

$$y = x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

1995B3. To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

1987B4. Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \ldots$. For each of $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

1997B5. Prove that for $n \geq 2$,

$$\left.2^{2^{\cdot^{\cdot^{2}}}}\right\}^{n} \equiv \left.2^{2^{\cdot^{\cdot^{2}}}}\right\}^{n-1} \pmod{n}.$$

1988B6. Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t the number $at + b$ is a triangular number if and only if t is a triangular number. (The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, \ldots\}$.)

This handout can be found at **http://math.stanford.edu/˜vakil/putnam06/**

E-mail address: vakil@math.stanford.edu