PUTNAM PROBLEM-SOLVING SEMINAR WEEK 5: INVARIANTS

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The Rules. These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

W1. (a) Can you tile an (8×8) chessboard with 32 (2×1) dominoes? (b) If you remove opposite corner squares from the chessboard, can you tile the remainder with 31 dominoes?

W2. *Sperner's lemma.* Divide a triangle into lots of baby triangles, so that baby triangles only meet at a common edge or a common vertex. Label each main vertex of the whole triangle by 1, 2, or 3; then label vertices on the (12) side by either 1 or 2, on the (23) side by either 2 or 3, and the (13) side by either 1 or 3. Label the points in the interior in any old way, by any of 1, 2, or 3. Show that any such labeling must contain a baby (123) triangle (in fact, an odd number of them). (Follow-up: how does this generalize to n dimensions? If you know what the Brouwer fixed point theorem is, then try to prove it using Sperner's lemma. This and other fun facts are on Harvey Mudd's "Math Fun Facts" website, `http://www.math.hmc.edu/funfacts/`.)

W3. Suppose 2006 red points and 2006 blue points are given in the plane, no three collinear (and pairwise distinct). Show that one can match up the red points and the blue points so that no two of the corresponding line segments intersect.

1. Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.

2. At first, a room is empty. Each minute, either one person enters or two people leave. After exactly 3^{2006} minutes, could the room contain $3^{1000} + 2$ people?

3. If 127 people play in a singles tennis tournament, prove that at the end of the tournament the number of people who have played an odd number of games is even.

4. Let $P_1, \ldots, P_{2007}$ be distinct points in the plane. Connect the points with the line segments $P_1P_2, P_2P_3, \ldots, P_{2006}P_{2007}, P_{2007}P_1$. Can one draw a line that passes through the interior of every one of these segments?

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5. Can a 10×10 square be tiled with 4×1 "quadrominoes"?

6.Three bugs start out at points $(0, 0)$, $(3, 0)$ and $(0, 2)$. They move one at a time, in any order. Each bug can only move in a direction parallel to the line containing the other two bugs. Can two of the bugs switch places while the third ends up where it started? Can they end up at $(1, 1)$, $(6, 2)$, $(3, 4)$?

7. Start with the set $\{3, 4, 12\}$. You are then allowed to replace any two numbers a and b with the new pair $0.6a - 0.8b$ and $0.8a + 0.6b$. Can you transform the set into $\{4, 6, 12\}$? (by Tom Rike)

8. A collection of n beetles, each black or white in color, is arranged in a line. On each move, a black beetle turns pink, emitting a chemical which causes its immediate neighbors to switch from black to white, or white to black (as appropriate). Already pink bugs are not affected. Under what starting conditions is it possible for all n bugs to turn pink?

9. *(The "Lights out" game)* Suppose $n \geq 2$ light bulbs are arranged in a row, numbered 1 through n. Under each bulb is a button. Pressing the button will change the state of the bulb above it (from on to off or vice versa), and will also change the neighbors' states. (Most bulbs have two neighbors, but the bulbs on the end have only one.) The bulbs start off randomly (some on and some off). For which n is it guaranteed to be possible that by flipping some switches, you can turn all the bulbs off? (Follow-up: I used this problem on a midterm. Try to guess the course.)

10. The n cards of a deck (where n is an arbitrary positive integer) are labeled $1, 2, \ldots, n$. Starting with the deck in any order, repeat the following operation: if the card on top is labeled k, reverse the order of the first k cards. Prove that eventually the first card will be 1 (so no further changes occur).

11. Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

Sound's Problem of the Week. (Subtle political satire added by Ravi.) An indicted senator gives you an $m \times n$ chocolate bar. What is the minimum number of times you must break it in order to get 1×1 squares?

This handout can be found at **http://math.stanford.edu/˜vakil/putnam06/**

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