PUTNAM PROBLEM-SOLVING SEMINAR WEEK 3: ANALYSIS

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The Rules. These are way too many problems to consider in this evening session alone. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Things to remember.

Calculus. Riemann Sums: if a function is Riemann-integrable, e.g. if it is continuous on a closed finite interval, then the integral is the limit of the Riemann sums.

Continuity. Remember these via pictures! The Intermediate Value Theorem: If f is continuous on $[a,b]$, then every value between $f(a)$ and $f(b)$ is of the form $f(c)$ for $a \leq c \leq b$. The Extreme Value Theorem. (More generally and usefully: a continuous function on a compact set attains its sup and inf.) Rolle's Theorem: If f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a)=f(b)$, then there is a point $u \in (a,b)$ at which $f'(u)=0$. The Mean Value Theorem: If f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a)=f(b)$, then there is a point $u \in (a,b)$ at which $f'(u) = \frac{f(b)-f(a)}{b-a}$.

Convergence. Sequences that converge: a bounded monotone sequence converges. A sum in which the entries have alternating sign and which decrease in absolute value converge. A monotone sum whose corresponding integral is bounded converges (the integral comparison test). A sequence bounded above and below by two other convergent sequences must converge (the squeeze principle).

Inequalities of integrals: $f \leq g$ means $\int_a^b f \leq \int_a^b g$ if $a \leq b$. *Taylor's Formula with Remainder:* if h has continuous nth derivatives, then for any $x > 0$ and integer $n > 0$, there exists $\theta_n \in [0,x]$ such that

$$h(x) = h(0) + h'(0)x + \cdots + h^{(n-1)}(0)x^{n-1}/(n-1)! + h^{(n)}(\theta_n)x^n/n!.$$

How I remember them both: problem W4.

"Big O *and little* o *notation":* $O(g(n))$ is a stand-in for a function $f(n)$ for which there exists a constant C such that $|f(n)| \leq C|g(n)|$ for all sufficiently large n. (This does not

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necessarily imply that $\lim_{n\to\infty} f(n)/g(n)$ exists.) Similarly "$f(t) = O(g(t))$ as $t \to 0$" means that there exists a constant C such that $|f(t)| \leq C|g(t)|$ for sufficiently small nonzero t. $o(g(n))$ is a stand-in for a function $f(n)$ such that $\lim_{n\to\infty} f(n)/g(n) = 0$. One can similarly define "$f(t) = o(g(t))$ as $t \to 0$".

THE PROBLEMS:

W1. Compute $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}$.

W2. Show that right now, there are two diametrically-opposed points on the earth's equator that are exactly the same temperature. (Follow-up problems: Are there two points on the earth's equator separated by 120 degrees that are exactly the same temperature? How about π degrees? Are there two diametrically opposed points on the earth's surface with the same temperature and air pressure?)

W3. Suppose $f(x)$ is a polynomial of odd degree. Show that $f(x) = 0$ has a real root.

W4. Show that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

1. Recall integration by parts:

$$\int f\ dg = fg - \int g\ df.$$

Substitute $f(x) = 1/x$, $g(x) = x$, and manipulate, to get

$$\int \frac{1}{x}\ dx = 1 + \int \frac{1}{x}\ dx.$$

Hence $0 = 1$. What has gone wrong?

2. Prove Bernoulli's Inequality: $(1 + x)^a \geq 1 + ax$ for $x > -1$ and $a \geq 1$, with equality when $x = 0$.

3. Suppose f is differentiable on $(-\infty, \infty)$ and there is a constant $k < 1$ such that $|f'(x)| \leq k$ for all real x. Show that f has a fixed point.

4. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, $(n, m = 0, 1, 2, \ldots)$? (Putnam 1990A2)

5. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \qquad n = 1, 2, 3, \ldots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \ldots$. (Putnam 1992A4)

6. Make sense of

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}.$$

7. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

(Putnam 1996B2. Hint: See W1.)

8. What happens if you put a random positive number in your (infinite precision) calculator, and repeatedly the sequence of buttons "$1/x$", "$+$" and "1"? (In other words, what happens if you iterate $x \mapsto 1/x + 1$?)

9. Let f be continuous and monotonically increasing, with $f(0) = 0$ and $f(1) = 1$. Prove that

$$f(1/10) + f(2/10) + \cdots + f(9/10) + f^{-1}(1/10) + f^{-1}(2/10) + \cdots + f^{-1}(9/10) \leq 99/10.$$

(Leningrad Math Olympiad 1991)

This handout can be found at **http://math.stanford.edu/˜vakil/putnam06/**

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