PROBLEM-SOLVING MASTERCLASS WEEK 6

1. Find the smallest term in the sequence $\{a_n\}$ where $a_1 = 1993^{1994^{1995}}$, $a_{n+1} = a_n/2$ if a_n is even, and $a_n + 7$ if a_n is odd. (Dragos Oprea)

2. Find all rational x, y such that $x^y = y^x$. (Nathan Pflueger, Larson 3.3.25a)

3. Let $B(n)$ be the number of ones in the base two expression for the positive integer n. For example, $B(6) = B(110_2) = 2$ and $B(15) = B(1111_2) = 4$. Determine whether or not

$$e^{\sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)}}$$

is a rational number. (John Le, 1981B5)

4. Prove that

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} \frac{1}{\sqrt{(2N-k)(2N-k-1)}} = \ln 2.$$

(Ching Hua Lee)

5. For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy $(x^2 + y^2)^m = (xy)^n$. (Kiyoto Tamura, 1992A3)

6. You are given a circle of diameter 1 in the plane, and n infinite strips of paper of lengths $\ell_1, \ldots, \ell_n$ respectively. Show that you can cover the circle with the strips of paper if and only if $\sum \ell_i \geq 1$. (Ravi Vakil)

Date: Monday, November 13, 2006.