PROBLEM-SOLVING MASTERCLASS WEEK 5

1. If a and b are positive integers, show that if $(a+1)/b + b/a$ is an integer then it equals 3. (Dragos Oprea)

2. Find all pairs of real numbers (x, y) satisfying the system of equations

$$\begin{aligned} \frac{1}{x} + \frac{1}{2y} &= (x^2 + 3y^2)(3x^2 + y^2) \\ \frac{1}{x} - \frac{1}{2y} &= 2(y^4 - x^4). \end{aligned}$$

(Nathan Pflueger, 2001B2)

3. Let a and b be positive integers, and define numbers x_n by $x_0 = 1$ and $x_{n+1} = ax_n + b$ for all $n \geq 0$. Prove that for any choice of a and b, the sequence $(x_n)_{n\geq 0}$ contains infinitely many composite numbers. (Siddhartha, # 9 from last week's seminar, left over from the "hit list")

4. A plays the following game at a casino. The casino picks n numbers at random from an unknown distribution, with n known to A. The casino reads the numbers to A, one at a time, without maliciously reordering them. A's goal is to pick the largest number, but he must do so at the time when the number appears, i.e. when the casino displays a number A either accepts or rejects it, but once he has accepted a number he cannot accept any that follow it. What is A's best strategy and what are his chances of winning? (Bob Hough)

5. A projectile is fired from the top of a hemispherical hill of height H. Its initial speed was fixed at speed V, but its angle of firing was allowed to vary. The hill sits on a level plain (like Ayer's Rock). Find the minimum distance the projectile can land on the plain from the bottom of the hill. Express your answer in terms of the given parameters (H, V and $g = 9.81\,m/s^2$), and provide conditions where there can be no solution (i.e. when V is too small). Hint: picture the hill as (half of) an open ball, and note that the projectile must not hit the hill. (Ching Hua Lee)

6. Find all rational x, y such that $x^y = y^x$. (Nathan Pflueger, Larson 3.3.25a)

7. Let $B(n)$ be the number of ones in the base two expression for the positive integer n. For example, $B(6) = B(110_2) = 2$ and $B(15) = B(1111_2) = 4$. Determine whether or not

$$e^{\sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)}}$$

is a rational number. (Ravi Vakil, 1981B5)

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