**PUTNAM PROBLEM-SOLVING SEMINAR WEEK 2: NUMBER THEORY**

**The Rules.** These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.


Number theory notions to know: modular arithmetic, unique factorization, greatest common divisor, division algorithm, r, s relatively prime means that there are a and b such that ar + bs = 1, Chinese Remainder Theorem, positional notation, Wilson’s Theorem \((p - 1)! \equiv -1 \pmod{p}\), Fermat’s little theorem \(a^p \equiv a \pmod{p}\), Euler \(\phi\)-function \(\phi(n) = \# \text{ numbers between 1 and } n \text{ relatively prime to } n\), \(a^{\phi(n)} \equiv 1 \pmod{n}\) if \(\gcd(a, n) = 1\).

1. Is a natural number uniquely determined by the product of its (positive) divisors? (Mark Lucianovic)

2. If \(2n + 1\) and \(3n + 1\) are both perfect squares, show that \(n\) is divisible by 40.

3. (a) How many zeros does \(1000!\) end with? (b) Is \(\binom{100}{36}\) even or odd? (Follow-up question: for how many \(k\) is \(\binom{100}{k}\) odd? The answer is surprising...)

4. Prove that \(\frac{a+b}{c+d}\) is irreducible if \(ad - bc = 1\).

5. (a) Show that there are an infinite number of primes of the form \(6n - 1\). (Hint: if there are only a finite number \(p_1, \ldots, p_k\), consider \((p_1 \cdots p_k)^2 - 1\).)
   (b) Prove that there are an infinite number of primes of the form \(4n - 1\).

6. Let \(n\) be a positive integer. Suppose that \(2^n\) and \(5^n\) begin with the same digit. Then there is only one possible value for this common initial digit. Find, with proof, that digit. (Sam Vandervelde, from Engel’s book *Problem-Solving Strategies*)

7. What are the last two digits of \(3^{1234}\)?

8. An (ordered) triple \((x_1, x_2, x_3)\) of positive irrational numbers with \(x_1 + x_2 + x_3 = 1\) is called balanced is \(x_i < 1/2\). If a triple is not balanced, say if \(x_j > 1/2\), one performs the

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following “balancing act”:  
\[ B(x_1, x_2, x_3) = (x'_1, x'_2, x'_3), \]

where \( x'_i = 2x_i \) if \( i \neq j \) and \( x'_j = 2x_j - 1 \). If the new triple is not balanced, one performs the balancing act on it. Does continuation of this process always lead to a balanced triple after a finite number of performances of the balancing act? (Putnam 1977)

9. Consider two lists. List A consists of the positive powers of 10 (10, 100, 1000, \ldots) written in base 2. List B consists of the positive powers of 10 written in base 5. Show that, for any integer \( n > 1 \), there is exactly one number in exactly one of the lists that is exactly \( n \) digits long.

<table>
<thead>
<tr>
<th>Powers of 10</th>
<th>List A</th>
<th>List B</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1010 (4 digits)</td>
<td>20 (2 digits)</td>
</tr>
<tr>
<td>100</td>
<td>1100100 (7 digits)</td>
<td>400 (3 digits)</td>
</tr>
<tr>
<td>1000</td>
<td>1111101000 (10 digits)</td>
<td>13000 (5 digits)</td>
</tr>
<tr>
<td>10000</td>
<td>10011100010000 (14 digits)</td>
<td>310000 (6 digits)</td>
</tr>
</tbody>
</table>

(1994 Asian Pacific Mathematical Olympiad)

10. For each positive integer \( n \), write the sum \( \sum_{m=1}^{n} \frac{1}{m} \) in the form \( \frac{p_n}{q_n} \), where \( p_n \) and \( q_n \) are relatively prime positive integers. Determine all \( n \) such that 5 does not divide \( q_n \). (Putnam 1997 B3)

This handout can be found at [http://math.stanford.edu/~vakil/putnam05/](http://math.stanford.edu/~vakil/putnam05/)

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