PROBLEM-SOLVING MASTERCLASS WEEK 7

We'll end with an "all-star" session of seven problems.

1. Show that
$$\pi \cot(\pi x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{x+n}.$$
(Theo Johnson-Freyd, from *Proofs from the Book*, which is an amazing book)

2. You have coins $C_1, C_2, \dots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n. (Cihan Baran, Putnam 2001A2)

3. Let G_n be the geometric mean of the binomial co-efficients in the nth row of Pascal's triangle $\binom{n}{0}, \dots, \binom{n}{n}$. Find $\lim_{n\to\infty} G_n^{1/n}$. (Bob Hough, from Andreescu and Galcea's book *Putnam and Beyond*)

4. If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum
$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$
of binomial coefficients is divisible by p^2. (Nathan Pflueger, Putnam 1996A5)

5. For positive a, b, and c, show that
$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a+b+c)^3.$$
(Kiyoto Tamura, likely from a recent USAMO)

6. The sequence u_n is defined by $u_0 = 1$, $u_{2n} = u_n + u_{n-1}, u_{2n+1} = u_n$. Show that for any positive rational k we can find n such that $\frac{u_n}{u_{n+1}} = k$. (Kiat Chuan Tan, Putnam 2002A5)

7. *Pick's theorem!* The area of any (not necessarily convex) polygon $Q \subset \mathbb{R}^2$ with integral vertices is given by:
$$A(Q) = n_{int} + \frac{1}{2} n_{bd} - 1,$$
where n_{int} is the number of integral points in the interior of Q and n_{bd} is the number of integral points on the boundary of Q. (Woodley Packard)

E-mail address: `vakil@math.stanford.edu`

Date: Monday, November 28, 2005.