PUTNAM PROBLEM-SOLVING SEMINAR WEEK 6: JUGGLING

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Juggling problems!

1. (The Average Theorem)

(a) The number of balls necessary to juggle a juggling sequence equals its average.
(b) Let j be a juggling function. If $\mathtt{height}(j)$ is finite, then

$$\lim_{|I|\to\infty} \frac{\sum_{i\in I} j(i)}{|I|}$$

exists, is finite, and is equal to $\mathtt{balls}(j)$, where the limit is over all integer intervals $I = \{a, a+1, a+2, \ldots, b\} \subset \mathbb{Z}$, and $|I| = b - a + 1$ is the number of integers in I.

2. ("Converse" of The Average Theorem — HARD!) Given a finite sequence of nonnegative integers whose average is an integer, there is a permutation of this sequence that is a juggling sequence.

Putnam problems!

Here are 18 problems from recent Putnams. Keep in mind that getting somewhere on a single problem is an achievement. Getting a problem in its entirety will usually place you well above the median. There are very few part marks. This should affect your strategy. Hence when you get a problem, *be sure to write it up very well*; don't scribble something down and then jump to another problem. Also, the problems tend to increase in difficult from 1 to 6, and the B's tend to be a touch harder than the A's. But there are always (unintentionally) higher-numbered problems that aren't so hard, and lower-numbered problems that are killers. Furthermore, you may be able to nail a high-numbered problem because you've seen some fact that others haven't. So don't write them off!

1998A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Date: Monday, November 15, 2004.

1998A2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

1998A3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

1998B1. Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

1998B2. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

1998B3. Let H be the unit hemisphere $\{ (x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0 \}$, C the unit circle $\{ (x, y, 0) : x^2 + y^2 = 1 \}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α, β are real numbers.

1999A1. Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that, for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

1999A2. Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x), \ldots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

1999A3. Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

1999B1. Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0}|EF|$. [Here $|PQ|$ denotes the length of the line segment PQ.] (I will draw the figure on the board.)

1999B2. Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots. [The roots may be either real or complex.]

1999B3. Let $A = \{(x,y) : 0 \le x, y < 1\}$. For $(x,y) \in A$, let

$$S(x,y) = \sum_{\frac{1}{2}\le\frac{m}{n}\le 2} x^m y^n,$$

where the sum ranges over all pairs (m,n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{\substack{(x,y)\to(1,1)\\(x,y)\in A}} (1-xy^2)(1-x^2y)S(x,y).$$

2000A1. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

2000A2. Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of two squares of integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

2000A3. The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5 and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

2000B1. Let a_j, b_j, c_j be integers for $1 \le j \le N$. Assume, for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \le j \le N$.

2000B2. Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \ge m \ge 1$.

2000B3. Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeros[1] (including multiplicities) of $\frac{d^kf}{dt^k}$. Prove that

$$N_0 \le N_1 \le N_2 \le \cdots \text{ and } \lim_{k\to\infty} N_k = 2N.$$

This handout can soon be found at **http://math.stanford.edu/~vakil/putnam04/**

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[1]The proposers intended for N_k to count only the zeros in the interval $[0,1)$.