PUTNAM PROBLEM-SOLVING SEMINAR WEEK 5: COUNTING

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Connect your problem with other counting problems. Count the complement in the set. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

1. (a) How many ways can you choose 12 cans of soup from among five different varieties if the order chosen doesn't matter?

(b) Same question, but now the order DOES matter.

2. How many odd 4-digit numbers contain at least one even digit?

(b) How many numbers less than 1,000,000 have a 2 in their decimal expansion?

3. Determine the number of different pizzas that can be made from a choice of 9 toppings.

4. Given the set $\{1, 2, \ldots, n\}$, find all permutations not containing the strings 124, 25, 35, or 213.

5. For $n \geq 0$, prove

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

6. How many different nonsense words (i.e. strings of letters) can be made by permuting the letters of "optimization"?

7. Prove the following identity:

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}(2^{n+1} - 1).$$

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(b) Prove that for every $n \geq 1$

$$\sum_{k=0}^{n} (2k+1)\binom{n}{k} = (n+1)2^n.$$

8. How many ways can 9 police officers be grouped into teams of 3?

9. The Stirling number $S(n,k)$ is the number of ways to put n distinguished balls into k indistinguishable cells, with no empty cells allowed. Find a closed formula for $S(n, n-3)$ as a function of n.

(b) Show that

$$S(n,k) = kS(n-1,k) + S(n-1,k-1).$$

10. Find the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 15$$

if $0 \leq x_1, x_2 \leq 6$ and $1 \leq x_3, x_4 \leq 7$.

11. A social worker has to make a total of 43 visits, at least one a day, over the course of 22 days. Is there a period of consecutive days in which she makes exactly 21 visits?

(b) Same question, but now over the course of 23 days.

12. A bit string (of ones and zeros) has even parity if it has an even number of 1's. How many bit strings of length n have even parity.

13. Prove the following identity:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

14. (Found by Brian Munson, used in a paper by M. Ginzburg, "Some Immersions of Projective Space in Euclidean Space," *Topology*,1963) Prove that for all positive n,

$$n! = 2^{n-\alpha(n)} \times (\text{odd number})$$

where $\alpha(n)$ is the number of 1's in the binary expansion of n. (How about other primes?)

(b) A number k is said to "fit" n if the location of 1's in the binary expansion of k is a subset of the location of 1's in n. Prove that k fits n if and only if $\binom{n}{k}$ is odd.

15. (Putnam 2000, B2) Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

This handout can soon be found at **http://math.stanford.edu/~vakil/putnam04/**

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