PUTNAM PROBLEM-SOLVING SEMINAR WEEK 4: THERE'S NO SUCH THING AS DEMOCRACY!

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

1. In 1956, the U.S. House of Representatives voted on a bill calling for federal aid for school construction. An amendment was proposed stipulating that federal aid would be provided only to those states whose schools were integrated. The voters were loosely divided into three equal-sized interest groups: Republicans, northern Democrats, and southern Democrats. Of the three options available, the three groups had differing opinions. The Republicans favored no bill at all, but if one were to pass they preferred one restricting aid to integrated states. The southern Democrats, being from states with segregated schools, preferred the original bill first, no bill second, and the amended bill third. The northern Democrats preferred the amended bill, followed by the original bill and no bill.

Clearly the original bill would have passed, as the Democrats together preferred the original bill to no bill at all. However, in keeping with House procedure, the first vote was on whether to accept the amendment; the Republicans and northern Democrats together ensured that the amendment passed, because they both preferred the amended bill to the original bill. The second vote was whether to approve the amended bill, or to have no bill at all. This time, the Republicans and the southern Democrats, both preferring to have no bill rather than the amended bill, conspired to ensure the bill's defeat. Paradoxically, the proposal of a popular amendment to a popular bill ensured the bill's eventual defeat! How did this happen? Who made a mistake?

2. San Francisco has an instant-run-off voting system. Each voter lists the candidates in order preference. If one of the candidates has a majority of first-place votes, they are declared the winner. Otherwise, the candidate with the fewest first-place votes is removed from the ballot, and the process continues. How can this give counterintuitive results? In other words, how can it happen that the "obviously wrong" person might win, and the "obviously right" person might lose?

Date: Monday, November 1, 2004.

3. Batter A has a higher batting average than batter B for the first half of the season *and* A also has a higher batting average than B for the second half of the season. Does it follow that A has a better batting average than B for the whole season?

4. *(Do not try this at home!)* Three players, A, B, and C, are involved in a gunfight. They take turns shooting one bullet at a time until only one player remains. C is a perfect shooter; he kills his target every time. B has an accuracy of two kills in three attempts. Finally, A only hits his target one out of every three times. To be fair, it is decided that A should begin the gunfight, to be followed by B and then C (if they are alive), and then back to A, and so on. What should A do, and why?

5. You throw a dart at a square target. You win if you hit a point nearer the center than the edge. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that you win. Express your answer in the form $(a\sqrt{b} + c)/d$, where a, b, c, d are positive integers.

6. *Non-commutative dice.* Explain how to number three dice (called A, B, and C) so that on average A will score higher than B, B will score higher than C, and C will score higher than A. (One possible solution is on the desk in the front of the room. Easier question: show that this solution works.)

7. *Crazy dice.* Devise a pair of dice, cubes with positive integers on their faces, with exactly the same outcomes as ordinary dice (the sum 2 comes out once, the sum 3 comes out twice, etc.), but which are not ordinary dice. (*Crazy hint:* $(x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4+x^5+x^6)$)

8. There are n toothpicks in a row. The first player can take up to $n-1$ toothpicks. The second player can take at most the number of toothpicks that the first has just taken. The first can then take at most the number of toothpicks that the second has just taken. This continues until all the toothpicks are gone. The player who takes the last toothpick wins. For which n does the first player have a winning strategy? (Hint: Try to find out if the first player can force a win with $n = 1, 2, 3, 4, 5$. If the first player can force a win, what is her winning first move?)

9. *Safety deposit box.* A village stores its priceless villagely artifacts in a safety deposit box. It is important that there is no unauthorised access to this box, but it is equally important that the contents can be retrieved when necessary. Various subsets of the (finite) set of villagers are deemed to constitute a quorum. Village rules allow the box to be opened when a quorum is present, and not otherwise. Design a system of locks and keys which will enforce this, for any given collection of subsets. [For example, the rules might state that k villagers have to be present; or that either the village chief or two of his sons must be present.]

10. *Hats problem.* Seven prisoners are subjected to the following twisted game. Each prisoner has a hat placed on his/her head. The colour of each hat is chosen to be red or green by random coin-flip, but the prisoner has no way of knowing which it is. At the appointed hour, the seven prisoners are seated around a table so that each can see the colours of the hats on the other prisoners. Each prisoner must then cast a vote, in the form of a written declaration "I think there are ? green hats". If all seven prisoners unanimously vote for the correct number of hats, then they all go free. Otherwise, they all remain in captivity. The one ray of hope is that (before the game is played) the seven prisoners are allowed to confer and decide upon a voting strategy. How are they to maximise the probablility of escape, and what is that maximum probability?

11. *Generalized Borromean rings.* The Borromean rings is a famous configuration of three loops which cannot be separated, but if any one of the loops is cut and removed, then the remaining two loops *can* be separated. **(i)** For every positive integer $n \geq 2$, find a configuration of n loops in space with the property that the loops cannot be separated, but if any one loop is cut and removed then all the remaining loops can be completely separated from each other. **(ii)** For every pair of positive integers $n > k \geq 1$ find a configuration of loops in space such that if any k of the loops are cut and removed, then all of the remaining loops can be separated from each other; but if only $k-1$ loops are cut and removed, then the remaining loops cannot be completely separated. **(iii)** Generalise this along the lines of the safety depost box question.

12. Players $1, 2, 3, \ldots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

13. Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players, A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

14. *Colonel Blotto's game.* Colonel Blotto and his opponent each have 100 divisions, and are going to fight over 10 pieces of territory (regions). They therefore (independently) divide their forces up into 10 parts and send each part to one region. Ten fights take place, and the one with the larger force wins the region (or there may be a stand-off). The winner of the battle is the one with the most won territory. Make a strategy for the game (i.e. choose a list of 10 non-negative integers adding to 100); we'll have a Blotto tournament at the end of the evening, and see whose strategy is the best.

This handout can soon be found at

http://math.stanford.edu/~vakil/putnam04/

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