PUTNAM PROBLEM-SOLVING SEMINAR WEEK 3: RECURSION

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

1. (a) Let $\{a_n\}_{n=0}^{\infty}$ be a sequence with $a_0 = 2$, $a_1 = 5$, and

$$a_{n+2} = 5a_{n+1} - 6a_n.$$

Find a closed form for a_n.

(b) Let $\{b_n\}_{n=0}^{\infty}$ be a sequence with $b_0 = 0$, $b_1 = 2$, and

$$b_{n+2} = 4b_{n+1} - 4b_n.$$

Find a closed form for b_n. (This is trickier than (a)!)

2. What is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$?

3. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 \neq 0$ and

$$a_{n+3} = 2a_{n+2} + 5a_{n+1} - 6a_n.$$

Find all possible values for $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

4. Suppose $a_n = 2^n + 3^{n+1}$. Find a recursion satisfied by a_n.

5. Suppose $a_n = -a_{n-1} - a_{n-2}$, and $a_0 = 100$. Find a_{2004}. (There doesn't seem to be enough information here!)

6. The sequence $r_1, r_2, \ldots$ satisfies $r_n = (5/2)r_{n-1} - r_{n-2}$, and $r_1 = 2004$. Suppose the sequence converges to a finite real number. Find r_2.

7. The sequence $G_0, G_1, G_2, \ldots$ consists of every other Fibonacci number. Show that there is a linear recursion (e.g. of the form $G_n = aG_{n-1} + bG_{n-2}$.) (Follow-up: How

Date: Monday, November 1, 2004.

about a sequence consisting of every *tenth* Fibonacci number. How do you know there's a recursion? Harder: With integer co-efficients?)

8. Find a length two recurrence satisfied by $c_n = \cos n^\circ$.

9. An elf skips up a flight of numbered stairs, starting at step 1 and going up one or two steps with each leap. He counts how many ways he can reach the nth step, calling the nth number E_n. What is E_n?

10. A gambling graduate student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring exactly n points at some time in a sequence of n tosses is $(2 + (-1/2)^n)/3$. (*Hint:* Let P_n denote the probability of scoring exactly n points at some time. Express P_n in terms of P_{n-1}, or in terms of P_{n-1} and P_{n-2}. Use this linear recursion to give an inductive proof. Even better hint, useful in many circumstances: you've been given the answer, so reverse-engineer the recursion, and then try to prove it.)

11. Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

12. Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m, \quad \text{and} \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

13. Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1 \text{ for } n = 1, 2, 3, \ldots.$$

Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

14. For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $\lim_{n\to\infty} d_n = \infty$.

Sample recurrence write-up.

Problem. Solve the linearly recurrent equation $f_n = 3f_{n-2} + 2f_{n-3}$, with initial conditions $f_0 = 1$, $f_1 = 1$, $f_2 = 6$.

Solution. The characteristic equation for this recurrence is $t^3 - 3t - 2 = 0$, i.e. $(t+1)^2(t-2) = 0$. The solutions are $t = -1$ (with multiplicity 2) and $t = 2$ (with multiplicity 1). Thus the solutions are all of the form $(An + B)(-1)^n + C2^n$. Using the values at $n = 0$, 1, and 2, we see that $A = 1$, $B = 0$, $C = 1$, and that the solution is $n(-1)^n + 2^n$.

This handout can soon be found at

http://math.stanford.edu/~vakil/putnam04/

E-mail address: lng@math.stanford.edu, vakil@math.stanford.edu