PUTNAM PROBLEM-SOLVING SEMINAR WEEK 2: NUMBER THEORY AND MODULAR ARITHMETIC

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Try induction or pigeonhole principle. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Number Theory in a Nutshell

- We write $b|a$ ("b divides a") iff there exists an integer q with $a = qb$.
- The greatest common divisor of a and b is denoted $\gcd(a, b)$.
- If $b > 0$ then there exist unique integers q, r with $a = qb + r$ and $0 \le r < b$. Fact: $\gcd(a, b) = \gcd(b, r)$. Repeating the division process to find the $\gcd$ is called the *Euclidean Algorithm*.
- There exist integers m,n such that $am + bn = \gcd(a, b)$. (They can be found by working backwards through the Euclidean Algorithm.) If there exist integers m,n such that $am+bn = c$ then $\gcd(a, b)|c$. In particular, if $am+bn = 1$ then $\gcd(a, b) = 1$, and a and b are *relatively prime*.
- The equation $ax + by = c$ has a solution iff $g = \gcd(a, b)$ divides c. Furthermore, if (x_0, y_0) is a solution then all solutions are given by $x = x_0 + bk/g, y = y_0 - ak/g$, where k is any integer.
- A positive integer n can be *uniquely* factored into primes as $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. This number has $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$ distinct factors.
- Euler's $\phi(n)$ function counts how many positive integers between 1 and n are relatively prime to n. Fact: if $\gcd(n_1, n_2) = 1$ then $\phi(n_1 n_2) = \phi(n_1)\phi(n_2)$. If n has unique prime factors $p_1, p_2, \ldots, p_k$ then $\phi(n) = n(1 - 1/p_1)(1 - 1/p_2) \cdots (1 - 1/p_k)$.
- We write $a \equiv b \pmod n$ ("a is congruent to b mod n") if $n|(a - b)$.
- If $a \equiv b \pmod n$ and $b \equiv c \pmod n$ then $a \equiv c \pmod n$.
- If $a \equiv b$ and $c \equiv d \pmod n$ then $a + c \equiv b + d$ and $ac \equiv bd \pmod n$.
- There exists an integer b such that $ab \equiv 1 \pmod n$ iff $\gcd(a, n) = 1$. This b is the *inverse* of a mod n, and can be computed from the Euclidean Algorithm.
- Euler: $a^{\phi(n)} \equiv 1 \pmod n$ if $\gcd(a, n) = 1$. Fermat: $a^p \equiv a \pmod p$ if p prime.
- Chinese Remainder Theorem. If $n_1, n_2, \ldots, n_k$ are pairwise relatively prime, then the system of congruences $x \equiv a_i \pmod{n_i}$, $i = 1, 2, \ldots, k$, has a *unique* solution x, mod $n_1 n_2 \cdots n_k$.

Date: Monday, October 25, 2004.

The Problems

1. You walk to a stream with a 5-pint container and a 7-pint container. How can you measure exactly 1 pint of water?

2. Do there exist 2004 consecutive integers such that each is divisible by a perfect cube bigger than 1?

3. Find the largest integer that is equal to the product of its digits.

4. Find all primes of the form $n^4 + 4$.

5. Prove that $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is irreducible for all positive integer n.

6. (a) Find the last two digits of 2^{2004}, 3^{2004}, and 6^{2004}. (b) How would you find the *first* two digits of the same numbers? You can use a standard calculator if needed.

7. Prove that there is a multiple of 3^{2004} that

(a) contains only the digits 0 and 1.

(b) contains all the digits 0, 1, $\ldots$, 9 at least once.

(c) ends in 2004.

8. Prove that if $F(x)$ is a polynomial with integral coefficients, and none of the integers $F(1)$, $F(2)$, $\ldots$, $F(2004)$ is divisible by 2004, then $F(x)$ has no integral zero.

9. Show that if n divides one Fibonacci number then it will divide infinitely many of them. (The Fibonacci numbers $\{f_k\}$ are $1, 1, 2, 3, 5, 8, \ldots$ where $f_k = f_{k-1} + f_{k-2}$.)

10. A *lattice point* is a point whose coordinates are both integers. Suppose we draw a line segment from $(0, 2004)$ to (a, b) where a and b are integers each randomly selected from $1, 2, 3, \ldots, 100$. What is the probability that the line segment contains an even number of lattice points?

11. (a) Show that the sequence $11, 111, 1111, 11111, \ldots$ contains no perfect squares. (b) Prove that there are no integers x and y for which $x^2 + 3xy + y^2 = 122$.

12. Prove that the expression

$$\frac{gcd(m, n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

13. Find all integers n such that $2^8 + 2^{11} + 2^n$ is a perfect square.

14. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. Find all real solutions to

$$\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 8x \rfloor = 2004.$$

(What happens if 2004 is replaced by 2003 or 2005?)

15. Show that there are infinitely many primes of the form $6n - 1$.

16. When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B.

17. Let $a_0 = 1$ and $a_{n+1} = a_n^2 + 1$ for $n \geq 0$. Find $\gcd(a_{2004}, a_{999})$.

18. Let n be a given positive integer. How many solutions are there in ordered positive-integer pairs (x, y) to the equation

$$\frac{xy}{x+y} = n?$$

19. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits of infinitely many a_i?

20. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

This handout can (soon) be found at

http://math.stanford.edu/~vakil/putnam04/

E-mail address: alok@kailas.stanford.edu (Alok Aggarwal), vakil@math.stanford.edu (Ravi Vakil)