PUTNAM PROBLEM-SOLVING SEMINAR WEEK 1: INDUCTION AND THE PIGEONHOLE PRINCIPLE

The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on. If you would like to practice with the Pigeonhole Principle or Induction (a good idea if you haven't seen these ideas before), try those problems.

The Hints. Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Induction problems.

Let a be an integer, and $P(n)$ a proposition (statement) about n for each integer $n \geq a$. The principle of mathematical induction states that if

(i) $P(a)$ is true, and
(ii) for each integer $k \geq a$, $P(k)$ true implies $P(k+1)$ true, then $P(n)$ is true for all integers $n \geq a$.

This principle enables us, in two simple steps, to prove an *infinite* number of propositions. It works best when you have observed a pattern and want to prove it.

1. Prove that $1 + 2 + \cdots + n = n(n+1)/2$.

2. Prove that the sum of the entries in the nth row of Pascal's triangle is 2^n (where the top row is "row 0").

3. Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet in a point (e.g. $f(4) = 11$). Find a formula for $f(n)$.

4. Find a formula for the sum of the first n odd numbers.

5. Show that for $n \geq 6$ a square can be dissected into n smaller squares, not necessarily all of the same size.

6. Show that $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = n(4n^2 - 1)/3$.

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7. If $a > 0$ and $b > 0$, then show that $(n-1)a^n + b^n \geq na^{n-1}b$, n a positive integer, with equality if and only if $a = b$.

8. (a) Prove that $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$. (b) Prove that $2!4!\cdots(2n)! \geq ((n+1)!)^n$.

9. Prove that $n^5/5 + n^4/2 + n^3/3 - n/30$ is an integer for $n = 0, 1, 2, \ldots$.

10. Prove the *arithmetic mean - geometric mean inequality* (AM-GM): Suppose $a_1, \ldots, a_n$ are n positive real numbers. Then

$$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 \cdots a_n)^{1/n}.$$

(Call this statement AMGM(n).) Prove AMGM(n) for all n as follows.

(a) Prove it for $n = 1$ and $n = 2$.
(b) If it is true for $n = k$, prove that it is true for $n = k - 1$. (*Hint:* substitute $a_n = (a_1 + \cdots + a_{n-1})/(n-1)$ in AMGM(n) and see what happens.)
(c) If it is true for $n = k$, prove that it is true for $n = 2k$.
(d) Conclude!

11. If each person, in a group of n people, is a friend of at least half the people in the group, then it is possible to seat the n people in a circle so that everyone sits next to friends only.

12. Suppose n coins are given, named $C_1, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

Pigeonhole principle problems.

If $kn + 1$ objects ($k \geq 1$) are distributed among n boxes, one of the boxes will contain at least $k + 1$ objects.

13. Given any $n + 1$ distinct integers between 1 and $2n$, show that one of them is divisible by another. Is this best possible, i.e., is the conclusion still true for n integers between 1 and $2n$?

14. Consider any five points $P_1, \ldots, P_5$ in the interior of a square S of side length 1. Show that one can find two of the points at distance at most $1/\sqrt{2}$ apart. Show that this is the best possible.

15. Consider any five points $P_1, \ldots, P_5$ in the interior of an equilateral triangle T of side length 1. Show that one can find two of the points at distance at most $1/2$ apart. What if there were four points?

16. Show that there are two people in New York City, who are not totally bald, who have the exact same number of hairs on their head.

17. Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

18. Show that if there are n people in the party, then two of them know the same number of people (among those present).

19. (a) Prove that in any group of six people there are either three mutual friends or three mutual strangers.

(b) Seventeen people correspond by mail with one another — each one with all the rest. In either letters only three topics are discussed. Each pair of correspondents deals with only one of the topics. Prove that there are at least three people who write to each other about the same topic.

20. Prove that no seven positive integers, not exceeding 24, can have sums of all subsets different.

21. Suppose there are given nine lattice points (points with integral coordinates) in three-dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.

22. Let α be irrational, and for $n \geq 1$, let $a_n = (n\alpha \bmod 1) \in [0, 1)$. (In other words, a_n is the "decimal part" of $n\alpha$.) Let us explain why $\{ a_n : n = 1, 2, \ldots \}$ is dense in $[0, 1]$ when α is irrational.

23. Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume, for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

24. Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

25. Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area $\geq 5/2$.

This handout can (soon) be found at

http://math.stanford.edu/˜vakil/putnam04/

E-mail address: `vakil@math.stanford.edu`