PROBLEM-SOLVING MASTERCLASS WEEK 7

1. Prove that there exists a positive real a such that for each natural number n, $[a^n]$ and n have the same parity. ($[x]$ denotes the largest integer not exceeding x.) (Kiyoto Tamura)

2. We have positive real numbers $t_1, t_2, \ldots, t_n$, such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

Prove that for any i, j, k that satisfies $1 \leq i < j < k \leq n$, one can construct a triangle with sides t_i, t_j, and t_k. (Ivan Janatra, IMO2004 # 4)

3. In the paper D. H. Bailey, P. B. Borwein and S. Plouffe, "On the Rapid Computation of Various Polylogarithmic Constants," Mathematics of Computation, vol. 66, no. 218 (April 1997), pg. 903-913, the authors used an automated "integer relation algorithm" to discover the identity

$$\pi = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1 - x^8} dx.$$

(a) Prove that the identity is true.

(b) Use the identity to show that

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right).$$

This formula allows rapid computation of the n^{th} hexadecimal digit of π. (Alok Aggarwal; he may also discuss the *integer relation algorithm*, voted one of the top 10 algorithms of the last century)

4. An enemy submarine is somewhere on the number line (consider only the integers for this problem). It is moving at some rate (again, integral units per minute). You know neither its position nor its velocity. You can launch a torpedo each minute at any integer on the number line. If the submarine is there, you hit it and it sinks. You have all the time and torpedoes you want. You must sink this enemy sub — devise a strategy that is guaranteed to eventually hit the enemy sub. (Ravi Vakil, from William Wu's website http://www.ocf.berkeley.edu/~wwu/riddles/hard.shtml)

5. A single lightbulb flickers into life in the center of the room. 100 prisoners shade their eyes from the glare, then focus on the prison warden standing by the lightswitch, with a standard evil-puzzler's glint in his eye. He begins to speak: "In one hour, you will all be taken to your cells to be kept in solitary confinement, with no possibility of communication with any of your fellow inmates. Well, almost no possibility ... every night from now on, I will choose one of you at random, retrieve you from your cell, and take you this room, where you may see if the lightbulb is *on* or *off*, and you may turn

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it *on* or *off* as you wish." A murmur ripples around the room as the prisoners consider the prospect of having such an effect on their hitherto impotent and externally controlled existences.

"If at some point, I take you to this room and you believe that all 100 prisoners have been chosen and taken here at some time, then you may tell me this. If you are correct, I will free you all. If of course you are incorrect ... well let's say none of you will live to flip any more lightbulb switches in this world." He exists with a flourish of his cloak, thoughtfully leaving the lightbulb on. The prisoners are in the dark as to how to et free, but they are perfectly clear about wanting to be able to at least flip light switches into old age. So they must come up with a strategy that will announce that all 100 prisoners have been chosen only if they actually have, with 100 % certainty. (Daniel Ford, or possibly Paul-Olivier Dehaye or Henry Segerman, from their article in the *Mathematical Intelligencer*, see http://math.stanford.edu/~dford/100prisoners/prisoners.pdf)