PROBLEM-SOLVING MASTERCLASS WEEK 6

1. *Conclusion of proof of:* Find all functions f from the real numbers to the real numbers such that for every two real numbers x and y, the following equation holds:

$$f(xf(x) + f(y)) = f(x)^2 + y.$$

(2004 Japanese Mathematical Olympiad, Kiyoto Tamura)

2. Three spiders and a fly are moving along the edges of a regular tetrahedron. The three spiders want to catch the fly. Problem: The fly is invisible, very intelligent, and psychic. The rules are as follows:

i. The spiders have to devise a strategy beforehand, i.e./ a certain fixed path they will follow.

ii. The spiders choose their starting points.

iii. The fly knows the strategy of the spiders and will always choose the best path to avoid the spiders.

iv. The fly may choose its starting point (after the spiders have chosen theirs).

v. The spiders are minimally faster than the fly. Say, when the spiders have moved 1 edge, the fly could have moved only 999/1000 of an edge.

vi. The spiders have no way to "see" the fly, but if a spider runs over the fly they win.

Is there a way the spiders can eventually catch the fly for sure? If so, give a strategy/path. If not, prove that they can't. (Henry Segerman)

3. Suppose x and y are integer solutions of the equation

$$2x^2 + x = 3y^2 + y.$$

Prove that $x - y$ and $2x + 2y + 1$ are perfect squares. (Ivan Ivan Janatra)

4. (a) Prove that for all positive n,

$$n! = 2^{n-\alpha(n)} \times (\text{odd number})$$

where $\alpha(n)$ is the number of 1's in the binary expansion of n. (How about other primes?)

(b) A number k is said to "fit" n if the location of 1's in the binary expansion of k is a subset of the location of 1's in n. Prove that k fits n if and only if $\binom{n}{k}$ is odd. (From last weeks' problem set; found by Brian Munson, used in a paper by M. Ginzburg, "Some Immersions of Projective Space in Euclidean Space," in the journal *Topology*, 1963; Bob Hough)

5. Prove that there exists a positive real a such that for each natural number n, $[a^n]$ and n have the same parity. ($[x]$ denotes the largest integer not exceeding x.) (Kiyoto Tamura)

6. Evaluate $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})}\,dt$. You may assume that $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$. (1985B5, Alex Chen)

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