PROBLEM-SOLVING MASTERCLASS WEEK 5

1. Find all functions f from the real numbers to the real numbers such that for every two real numbers x and y, the following equation holds:

$$f(xf(x) + f(y)) = f(x)^2 + y.$$

(2004 Japanese Mathematical Olympiad, Kiyoto Tamura)

2. In each lattice square some real number is written. F1 and F2 are figures consisting of finite sets of unit squares. F1 has the property that for each translation (in any direction, but by a vector (x, y) where x and y are integers) the sum of the numbers in its translation is positive. Prove that F2 can also be translated so that the sum of the numbers written in its translation is positive. (2000 Romanian IMO training, Florin Ratiu)

3. (i) Suppose that neither x nor y are perfect squares modulo p (where p is prime). Show that xy *is* a perfect square modulo p. (ii) For which primes p is -1 a perfect square modulo p? (iii) (from the book "A Problem Seminar") Show that the number 16 is a perfect 8th power modulo p for any prime p. (Ravi Vakil)

4. Evaluate

$$\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}$$

(1980 Putnam, Alex Chen)

5. Devise a pair of dice, cubes with positive integers on their faces, with exactly the same outcomes as ordinary dice (the sum 2 comes out once, the sum 3 comes out twice, etc.), but which are not ordinary dice. (*Crazy hint:* $(x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4+x^5+x^6)$) ("A Problem Seminar", Ravi Vakil; on an earlier week's Putnam seminar handout)

6. Three spiders and a fly are moving along the edges of a regular tetrahedron. The three spiders want to catch the fly. Problem: The fly is invisible, very intelligent, and psychic. The rules are as follows:

i. The spiders have to devise a strategy beforehand, ie a certain fixed path they will follow.

ii. The spiders choose their starting points.

iii. The fly knows the strategy of the spiders and will always choose the best path to avoid the spiders.

iv. The fly may choose its starting point (after the spiders have chosen theirs).

v. The spiders are minimally faster than the fly. Say, when the spiders have moved 1 edge, the fly could have moved only 999/1000 of an edge.

vi. The spiders have no way to "see" the fly, but if a spider runs over the fly they win.

Is there a way the spiders can eventually catch the fly for sure? If so, give a strategy/path. If not, prove that they can't. (Henry Segerman)

Date: Monday, November 15, 2004.