PROBLEM-SOLVING MASTERCLASS WEEK 3

1. Let S be a set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that

(1) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;
(2) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$ (for a, b, c distinct);
(3) (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

Prove that there exists a one-to-one function g from A to $\mathbb{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$. (1996A4, John Hegeman)

2. Find all integer solutions to $a^2 + b^2 + c^2 = a^2 b^2$. (Kiyoto Tamura)

3. Let A_n be the number of elements in the nth row of Pascal's triangle that are congruent to 1 modulo 3, and let B_n be the number of elements that are congruent to 2 modulo 3. Show that $A_n - B_n$ is always a power of 2. (Ravi Vakil)

4. Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \geq 2$), of

$$x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1.$$

(1996B3, John Hegeman)

E-mail address: vakil@math.stanford.edu

Date: Monday, November 1, 2004.