PROBLEM-SOLVING MASTERCLASS WEEK 2

1. You have an infinite quarter plane chessboard, with three tokens in the corner:

   O O · · · ·
   O · · · ·
   · · · ·
   · · · ·

   You can make one sort of move: take a token, and replace it with 2 tokens, one directly to the right and one directly below the token you remove:

   O · · · · ⇒ · O
   · · · ·

   (You may never have two tokens in the same square.) Using this move, is it possible to clear the starting three squares? (Henry Segerman)

2. Let $a_1, a_2, \ldots a_n$ be $n$ positive real numbers, and $b_1, \ldots, b_n$ be $n$ distinct positive real numbers ($n \geq 2$). Let

   \[ S = a_1 + a_2 + \cdots + a_n \quad \text{and} \quad T = b_1 b_2 \cdots b_n. \]

   Show that

   \[ \frac{\sum_{j=1}^{n} b_j (1 - a_j / S)}{n - 1} > \left( \frac{T}{S} \sum_{j=1}^{n} \frac{a_j}{b_j} \right)^{\frac{1}{n - 1}} \]

   (2002 Korean Mathematical Olympiad, final round, Hae Kang Lee)

3. Show that for every positive integer $n$,

   \[ \left( \frac{2n - 1}{e} \right)^{\frac{2n - 1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n - 1) < \left( \frac{2n + 1}{e} \right)^{\frac{2n + 1}{2}}. \]

   (1996B2, John Hegeman)

4. The sequence of digits

   123456789101112131415161718192021 \ldots

   is obtained by writing the positive integers in order. If the $10^{th}$ digit in this sequence occurs in the part of the sequence in which the $m$-digit numbers are placed, define $f(n)$ to be $m$. For example, $f(2) = 2$ because the $100^{th}$ digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$. (1987A2, Alex Chen)

Date: Monday, October 25, 2004.
5. Prove that if \( n \) has at least two distinct prime divisors then there is some permutation, \( \phi \), of \( \{1, 2, \ldots, n\} \) such that
\[
\phi(1) \cos\left(\frac{2\pi}{n}\right) + \phi(2) \cos\left(\frac{4\pi}{n}\right) + \cdots + \phi(n) \cos\left(\frac{2\pi}{n}\right) = 0.
\]
(Bob Hough)

6. Let \( S \) be a set of ordered triples \((a, b, c)\) of distinct elements of a finite set \( A \). Suppose that

(1) \((a, b, c) \in S\) if and only if \((b, c, a) \in S\);

(2) \((a, b, c) \in S\) if and only if \((c, b, a) \notin S\) [for \( a, b, c \) distinct];

(3) \((a, b, c)\) and \((c, d, a)\) are both in \( S \) if and only if \((b, c, d)\) and \((d, a, b)\) are both in \( S \).

Prove that there exists a one-to-one function \( g \) from \( A \) to \( \mathbb{R} \) such that \( g(a) < g(b) < g(c) \) implies \((a, b, c) \in S\). (1996A4, John Hegeman)

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