PROBLEM-SOLVING MASTERCLASS WEEK 2

1. You have an infinite quarter plane chessboard, with three tokens in the corner:

$$\begin{array}{ccccc} O & O & \cdot & \cdot & \cdot & \cdot \\ O & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot \\ \cdot \end{array}$$

You can make one sort of move: take a token, and replace it with 2 tokens, one directly to the right and one directly below the token you remove:

$$\begin{array}{cc} O & \cdot \\ \cdot & \cdot \end{array} \quad \Rightarrow \quad \begin{array}{cc} \cdot & O \\ O & \cdot \end{array}$$

(You may never have two tokens in the same square.) Using this move, is it possible to clear the starting three squares? (Henry Segerman)

2. Let $a_1, a_2, \ldots a_n$ be n positive real numbers, and $b_1, \ldots, b_n$ be n distinct positive real numbers ($n \geq 2$). Let

$$S = a_1 + a_2 + \cdots + a_n \qquad \text{and} \qquad T = b_1 b_2 \cdots b_n.$$

Show that

$$\frac{\sum_{j=1}^{n} b_j(1 - a_j/S)}{n-1} > \left(\frac{T}{S} \sum_{j=1}^{n} \frac{a_j}{b_j} \right)^{\frac{1}{n-1}}$$

(2002 Korean Mathematical Olympiad, final round, Hae Kang Lee)

3. Show that for every positive integer n,

$$\left(\frac{2n-1}{e} \right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e} \right)^{\frac{2n+1}{2}}.$$

(1996B2, John Hegeman)

4. The sequence of digits

$$1234567891011121314151617181920 21 \ldots$$

is obtained by writing the positive integers in order. If the $10^{n\text{th}}$ digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100^{th} digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$. (1987A2, Alex Chen)

Date: Monday, October 25, 2004.

5. Prove that if n has at least two distinct prime divisors then there is some permutation, ϕ, of $\{1, 2, \ldots, n\}$ such that

$$\phi(1)\cos(2\pi/n) + \phi(2)\cos(4\pi/n) + \cdots + \phi(n)\cos(2\pi) = 0.$$

(Bob Hough)

6. Let S be a set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that

(1) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;
(2) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$ [for a, b, c distinct];
(3) (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

Prove that there exists a one-to-one function g from A to $\mathbb{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$. (1996A4, John Hegeman)

E-mail address: `vakil@math.stanford.edu`