PROBLEM-SOLVING MASTERCLASS WEEK 1

1. Given a finite string S of symbols X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example, $\Delta(\mathrm{XOOXOOX}) = -1$. We call a string S *balanced* if every substring T of (consecutive symbols of) S has $-2 \leq \Delta(T) \leq 2$. Thus, XOOXOOX is not balanced, since it contains the substring OOXOO. Find, with proof, the number of balanced strings of length n. (1996B5, John Hegeman)

2. Let a and b be two positive integers such that $ab \neq 1$. Find all integer values of

$$\frac{a^2 + ab + b^2}{ab - 1}.$$

(Romanian IMO training, Florin Ratiu)

3. Two people are walking randomly on the number line, each taking a step of length 1 every second, choosing whether to go left or right at random (with equal probability). What is the probability that, after N steps, they are in the same place? (Reif's *Statistical Mechanics*, Andy Lutomirski)

4. Show that if $0 < r < 1$ and if the complex numbers $z_1, z_2, \ldots, z_n$ are in the disk $D = \{z : |z| \leq r\}$, then there exists z_0 in D such that

$$(1 + z_1)(1 + z_2) \cdots (1 + z_n) = (1 + z_0)^n.$$

(Bob Hough)

5. The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the $10^{n\text{th}}$ digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100^{th} digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$. (1987A2, Alex Chen)

6. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

(1996B2, John Hegeman)

E-mail address: vakil@math.stanford.edu

Date: Monday, October 18, 2004.