1. Prove that for \( n \geq 2 \),
\[
2^2 \cdot 2 \cdot \cdots \cdot 2 \equiv 2^2 \cdot \cdots \cdot 2 \equiv 1 \pmod{n}.
\]
(Yuanli, Putnam 1997B5)

2a. Does there exist a function \( g : \mathbb{Z}^+ \to \mathbb{Z}^+ \) with the property that \( g(g(n)) = n + 1987 \) for all \( n \)? (Frank, IMO 1987)

2b. (Follow-up) Prove that \( f(n) = 1 - n \) is the only integer-valued function defined on the integers that satisfies the following conditions:

(i) \( f(f(n)) = n \), for all integers \( n \);
(ii) \( f(f(n+2)+2) = n \) for all integers \( n \);
(iii) \( f(0) = 1 \).

(Putnam 1992A1)

3a. Given any 5 distinct points on the surface of a sphere, show that we can find a closed hemisphere which contains at least 4 of them. (Andy, Putnam 2002A2)

3b. (Follow-up) A stick is broken at random in two places. What is the probability that the three pieces can form a triangle?

4. A function \( f \) is defined on the positive integers by
\[
\begin{align*}
f(1) &= 1, \\
f(3) &= 3, \\
f(2n) &= f(n) \\
f(4n+1) &= 2f(2n+1) - f(n) \\
f(4n+3) &= 3f(2n+1) - 2f(n)
\end{align*}
\]
for all positive integers \( n \). Determine the number of positive integers \( n \), less than or equal to 1988, for which \( f(n) = n \). (Shrenik, IMO 1988)

5. A rectangle is tiled with smaller rectangles. Each smaller rectangle has a side of integer length. Show that the same is true of the larger rectangle. (Ravi)

This handout can be found at

http://math.stanford.edu/~vakil/putnam03/

E-mail address: vakil@math.stanford.edu

Date: Monday, November 10, 2003.