

PUTNAM PROBLEM SOLVING SEMINAR WEEK 7

This is the last meeting before the Putnam.

The Rules. These are way too many problems to consider. Just pick a few problems in one of the sections and play around with them. The 1988 Putnam is left over from last week (so we can discuss the problems you thought about then).

Miscellaneous problems (at least some vaguely hat-related).

1. Understand the strategy in the 7-person hat game. If you are person number 6, and you see the following configurations, what should you do? (a) $BWBBB?B$. (Show that you will win in this case.) (b) $WBBWW?W$. (Will you win or lose?) (c) $BWWBB?W$.

2. (a) Show that the best strategy in the 8-person game will win at least $7/8$ of the time. (Hint: Give a strategy that will win *exactly* $7/8$ of the time.) (b) Show that as $n \rightarrow \infty$, the odds of winning (in the optimal strategy) tend to 1. (You can do this without knowing what the optimal strategy is in general!)

3. Find the optimal strategy in the following “hats-type” game. Once again, this is a cooperative game. Ten players are in a line (each facing forward, so the first person can’t see any of the others, the second can see only the first, etc.) The referee puts a hat on each player’s head, that is red, green, or blue (with equal probability). In some predetermined order, they guess their hat color. At the end, if all but one get their hat color right, they win a million dollars. Otherwise, they are all kicked in the shins.

The next few problems are variations of the “lights-out” games. Several lights are given (some on and some off), and each light has an attached button. When you press the button, the state of that light and the adjacent lights (where adjacent lights are defined below). In each case, the question is: Can you turn out all the lights, no matter the starting configuration?

4. There are four lights in a row, and “adjacent” means what you think it does. (Hint, for this and the rest of the problems. Show that the order in which you press the buttons is irrelevant; it only matters which buttons you press, and how often. Turn it into a problem about the vector space \mathbb{F}_2^4 .)

5. Same as in #4, except the four lights are in a circle.

6. Same as in #4, except with n lights in a row.

7. The lights are in a 3×3 array; adjacent means “horizontally, vertically, or diagonally adjacent”.

8. The lights are the vertices of a icosahedron; vertices are adjacent if they are connected by an edge. (I’ll have a model handy to play with.)

**The Forty-Ninth William Lowell Putnam Mathematical Competition
(December 3, 1988)**

A1. Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.

A2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b) .

A3. Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1 \right)^x$$

converges.

A4.

- (a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?
- (b) What if “three” is replaced by “nine”?

Justify your answers.

A5. Prove that there exists a *unique* function f from the set \mathbb{R}^+ of positive real numbers to \mathbb{R}^+ such that

$$f(f(x)) = 6x - f(x) \quad \text{and} \quad f(x) > 0 \quad \text{for all } x > 0.$$

A6. If a linear transformation A on an n -dimensional vector space has $n + 1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

B1. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y , and z positive integers.

B2. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y + 1) \leq (x + 1)^2$, then $y(y - 1) \leq x^2$.

B3. For every n in the set $\mathbb{Z}^+ = \{1, 2, \dots\}$ of positive integers, let r_n be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c + d = n$. Find, with proof, the smallest positive real number g with $r_n \leq g$ for all $n \in \mathbb{Z}^+$.

B4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

B5. For positive integers n , let \mathbf{M}_n be the $2n + 1$ by $2n + 1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1 . Find, with proof, the rank of \mathbf{M}_n . (According to one definition, the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

One may note that

$$\mathbf{M}_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

B6. Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t the number $at + b$ is a triangular number if and only if t is a triangular number.

(The triangular numbers are the $t_n = n(n + 1)/2$ with n in $\{0, 1, 2, \dots\}$.)

**The Fifty-Third William Lowell Putnam Mathematical Competition
(December 5, 1992)**

A1. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:

- (i) $f(f(n)) = n$, for all integers n ;
- (ii) $f(f(n + 2) + 2) = n$ for all integers n ;
- (iii) $f(0) = 1$.

A2. Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series expansion about $x = 0$ of $(1 + x)^\alpha$. Evaluate

$$\int_0^1 C(-y - 1) \left(\frac{1}{y + 1} + \frac{1}{y + 2} + \frac{1}{y + 3} + \dots + \frac{1}{y + 1992} \right) dy.$$

A3. For a given positive integer m , find all triples (n, x, y) of positive integers, with n relatively prime to m , which satisfy $(x^2 + y^2)^m = (xy)^n$.

A4. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2 + 1}, \quad n = 1, 2, 3, \dots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \dots$

A5. For each positive integer n , let

$$a_n = \begin{cases} 0 & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1 & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist positive integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j},$$

for $0 \leq j \leq m - 1$.

A6. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

B1. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S . For a given $n \geq 2$, what is the smallest possible number of elements in A_S ?

B2. For nonnegative integers n and k , define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n, k) = \sum_{j=0}^k \binom{n}{j} \binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.)

B3. For any pair (x, y) of real numbers, a sequence $(a_n(x, y))_{n \geq 0}$ is defined as follows:

$$\begin{aligned} a_0(x, y) &= x, \\ a_{n+1}(x, y) &= \frac{(a_n(x, y))^2 + y^2}{2}, \quad \text{for } n \geq 0. \end{aligned}$$

Find the area of the region $\{(x, y) | (a_n(x, y))_{n \geq 0} \text{ converges}\}$.

B4. Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no non-constant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}} \left(\frac{p(x)}{x^3 - x} \right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

B5. Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}.$$

Is the set $\{D_n/n!\}_{n \geq 2}$ bounded?

B6. Let \mathcal{M} be a set of real $n \times n$ matrices such that

- (i) $I \in \mathcal{M}$, where I is the $n \times n$ identity matrix;
- (ii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;
- (iii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB = BA$ or $AB = -BA$;
- (iv) if $A \in \mathcal{M}$ and $A \neq I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that \mathcal{M} contains at most n^2 matrices.

This handout, and other useful things, can (soon) be found at

<http://math.stanford.edu/~vakil/stanfordputnam.html>