

PUTNAM PROBLEM SOLVING SEMINAR WEEK 5

The Rules. These are way too many problems to consider. Just pick a few problems in one of the sections and play around with them.

The theory of games.

Explain who can win the following games by playing perfectly, and how. (Test your arguments by playing against other people.) Many involve *matchstick games*: Several piles of matches are given. Two players alternate playing. Each “play” involves removing a certain number of matches from one of the piles. The last person to play wins.

1. Each player can remove between one and three matches. The initial pile has 10 matches.
2. Solve the “misère” version of the previous game: the last person to play *loses*.
3. Each player can remove 2^n matches, for any non-negative integer n .
4. There are four piles of matches, with 7, 8, 9, and 10 matches respectively. Each player can remove between one and three matches from one of the piles.
5. There is one pile of matches. When the pile has n matches left, the next player may remove up to $2\sqrt{n+1} - 2$ matches. (The game ends when there is one match left.)
6. There are four piles of matches. If there are n matches in the pile, then the next player may remove 2^m matches, where 2^m appears in the binary representation of n .
7. Show that there is an “optimal strategy” for chess.
8. *The fifteen game.* (*This one is more a “trick” than a problem.*) Fifteen cards are on the table, numbered one through fifteen. The two players alternate picking up cards. The first player to have three cards summing to fifteen wins. If all cards are picked up without either player winning, the game is declared a “draw”. Show that (i) if both players play perfectly, the game will be drawn, and (ii) if one player “knows what’s going on”, she can do very well, for example by starting with any of the even cards. (Big hint for both parts of the problem: take a 3×3 magic square, and keep track of the picked-up cards there. Notice that you are really playing some other (famous) game.)

9. A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking **either**

- (a) one bean from a heap, provided at least two beans are left behind in that heap, **or**
- (b) a complete heap of two or three beans.

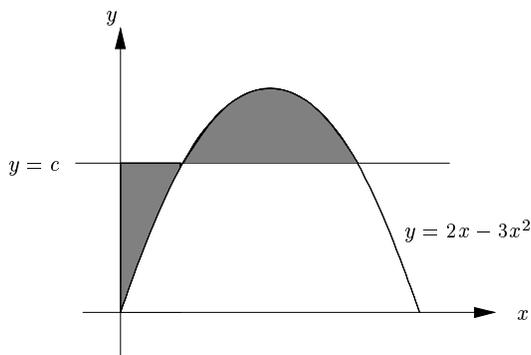
The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

10. See problem B2 below. (In the end, it isn't exactly a game theory problem.) One of the earlier problems also appeared on the Putnam; I'm not telling you which one, because you might think it's harder than it is!

The Fifty-Fourth William Lowell Putnam Mathematical Competition (December 4, 1993).

Caution: This was perhaps the hardest Putnam in the last couple of decades.

A1. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^2$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.



A2. Let $(x_n)_{n \geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1 \text{ for } n = 1, 2, 3, \dots$$

Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

A3. Let \mathcal{P}_n be the set of subsets of $\{1, 2, \dots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \rightarrow \{1, 2, \dots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^m j^n.$$

B6. Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y , where $x \leq y$ and replace them with $2x$ and $y - x$.

This handout, and other useful things, can (soon) be found at

<http://math.stanford.edu/~vakil/stanfordputnam.html>