# PUTNAM PROBLEM SOLVING SEMINAR WEEK 1

**The Rules.** You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on. If you would like practice with the Pigeonhole Principle or Induction (a good idea if you haven't seen these ideas before), try those problems.

**The Hints.** Try small cases. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backward. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. And ask!! If the problem has a 2001 in it, what happens if you replace 2001 by 1, or 2, or 3? What's important about 2001 — is it that it is odd, or divisible by 3, etc.?

**The Problems.**

*The next few problems are for practice with the pigeonhole principle.*

**1.** Prove that there are two people in the U.S. right now with the same amount of hair on their heads (not including bald people!).

**2.** Let $A$ be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, ..., 100. Prove that there must be two distinct integers in $A$ whose sum is 104.

**3.** Show that if there are $n$ people at a party, then two of them know the same number of people (among those present).

**4.** Five points lie in an equilateral triangle of size 1. Show that two of the points lie no farther than 1/2 apart. Can the "1/2" be replaced by anything smaller? Can it be improved if the "five" is replaced by "six"?

**5.** A lattice point in the plane is a point $(x, y)$ such that both $x$ and $y$ are integers. Find the smallest number $n$ such that given $n$ lattice points in the plane, there exist two whose midpoint is also a lattice point.

**6.** Prove that there is some integral power of 2 that begins 2001....

**7.** Prove that in any group of six people there are either three mutual friends or three mutual strangers. (Hint: Represent the people by the vertices of a regular hexagon. Connect two vertices with a red line segment if the couple represented

---

*Date*: October 9, 2001.

by these vertices are friends; otherwise, connect them with a blue line segment. Consider one of the vertices, say $A$. At least three line segments emanating from $A$ have the same color. There are two cases to consider.)

**8.** Follow-up to the previous question: Find some $n$ so that in any group of $n$ people there are either four mutual friends or four mutual strangers.

**9.** A polygon in the plane has area 1.2432. Show that it contains two distinct points $(x_1, y_1)$ and $(x_2, y_2)$ that differ by $(a, b)$, where $a$ and $b$ are integers.

*The next few problems are for practice with induction.*

**10.** Show that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

**11.** Show that for all positive integers $n$, $n^5/5 + n^4/2 + n^3/3 - n/30$ is an integer.

**12.** Show that $1 + 1/\sqrt{2} + 1/\sqrt{3} + \cdots + 1/\sqrt{n} < 2\sqrt{n}$.

**13.** Prove that all even perfect squares are divisible by 4. Prove that all odd perfect squares leave a remainder of 1 upon division by 8. (This is a useful fact to know!) What are the possible remainders when you divide a perfect square by 3?

*The last ten problems are not on any particular top; they are (very) roughly in increasing order of difficulty.*

**14.** (The handshake problem) Mr. and Mrs. Adams recently attended a party at which there were three other couples. Various handshakes took place. No one shook hands with his/her own spouse, no one shook hands with the same person twice, and of course, no one shook his/her own hand. After all the handshaking was finished, Mr. Adams asked each person, including his wife, how many hands he or she had shaken. To his surprise, each gave a different answer. How many hands did Mrs. Adams shake?

**15.** If $n$ is a positive integer such that $2n+1$ is a perfect square, show that $n+1$ is the sum of two successive perfect squares. If $3n+1$ is a perfect square, show that $n + 1$ is the sum of three perfect squares.

**16.** Fifteen pennies lie on the table in the shape of a triangle, with five pennies on each side. For some reason, the pennies are painted either black or white. Prove that there exist three pennies of the same color whose centers are the vertices of an equilateral triangle.

**17.** Let $x_1$, $x_2$, $x_3$, $\ldots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}},$$

for $n = 3, 4, 5, \ldots$. Establish necessary and sufficient conditions on $x_1$ and $x_2$ for $x_n$ to be an integer for infinitely many $n$.

**18.** The latest calculator has a button, which when pressed, replaces a number $x$ by $1+1/x$. You type in some positive number, and then start pressing the button repeatedly. What happens? Prove it! (The handout originally had $x+1/x$ in place of $1+1/x$. Many people noticed in this case that the numbers increase without bound. Again, prove it! This one is not as hard.)

**19.** Let $d$ be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m, \quad \text{and} \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

**20.** Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many $a_i$?

**21.** Let

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers $(m, n)$.

**22.** Let $C$ be the unit circle $x^2 + y^2 = 1$. A point $p$ is chosen randomly on the circumference of $C$ and another point $q$ is chosen randomly from the interior of $C$ (these points are chosen independently and uniformly over their domains). Let $R$ be the rectangle with sides parallel to the $x$- and $y$-axes with diagonal $pq$. What is the probability that no point of $R$ lies outside of $C$?

**23.** Let $a_1, a_2, \ldots, a_n$ be real numbers, and let $b_1, b_2, \ldots, b_n$ be distinct positive integers. Suppose there is a polynomial $f(x)$ satisfying the identity

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \ldots, b_n$ and $n$ (but independent of $a_1, a_2, \ldots, a_n$).

*This handout, and other useful things, can (soon) be found at*

**http://math.stanford.edu/~vakil/stanfordputnam.html**